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TOPOLOGICAL PROPERTIES AND LOCALIZED VIBRATION MODES IN QUASIPERIODIC METAMATERIALS WITH ELECTROMECHANICAL LOCAL RESONATORS

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ABSTRACT

Simultaneous energy harvesting and vibration attenuation has been a topic of great interest in many recent investigations in mechanical metamaterials. These studies have shown the ability to harvest electrical power using weak electromechanical coupling in periodic metamaterials with no effect on the material's bandgap boundaries. However, the effect of the electromechanical resonator on the topological properties (i.e. the bandgap topology) and localized mode shapes of a quasiperiodic metamaterial has not yet been determined. In this paper, we study a quasiperiodic metamaterial coupled to electromechanical resonators to observe its bandgaps and localized vibration modes. We show here the analytical dispersion surfaces of an infinite quasiperiodic metamaterial with electromechanical local resonators. The natural frequencies of a semi-infinite system are also simulated numerically to validate the analytical results and show the band structure for different quasiperiodic patterns, load resistors, and electromechanical coupling coefficients. Furthermore, the mode shapes are presented here for a semi-infinite structure showing localized vibration within the bandgaps. The results demonstrate that quasiperiodic metamaterials with electromechanical local resonators can be used to harvest energy without changing the topology of the bandgaps for the case of weak electromechanical coupling. The observations given here can be used to guide designers in choosing electromechanical resonator parameters and quasiperiodic pattern parameters for an effective energy harvesting metamaterial.

1 INTRODUCTION

Mechanical Metamaterials are a newer class of artificially structured materials that exhibit many unique dynamic properties [1]. These materials are of particular interest in the areas of vibration control and energy harvesting. Metamaterials with periodic structures, for example, are known to produce a bandgap in their frequency response for wavelengths close to their lattice constant [2–6]. In the frequency range of the bandgap, waves cannot propagate and are reflected back due to Bragg scattering.

Local resonators can also be imbedded in a structure to form locally resonant metamaterials with additional bandgaps at wavelengths much shorter than the lattice constant [7,8]. The location and size of these additional bandgaps can be tuned through design of the local resonator's parameters [9] and the number and location of resonators [10].

Recently, there has been great focus on the use of metamaterials for simultaneous vibration control and energy harvesting. By introducing piezoelectric patches to the resonators and shunting them to external circuits, these locally resonant metamaterials can be used to harvest electric power and control vibrations. [11–14]. It has been shown that weak electromechanical coupling does not alter the band structure or negatively impact the vibration reduction performance of locally resonant periodic metamaterials [15, 16]. Due to the electromechanical coupling

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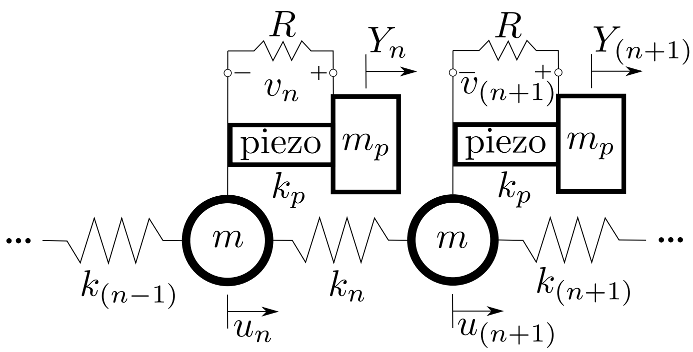

FIGURE 1. A SCHEMATIC OF THE QUASIPERIODIC METAMATERIAL WITH ELECTROMECHANICAL LOCAL RESONATORS

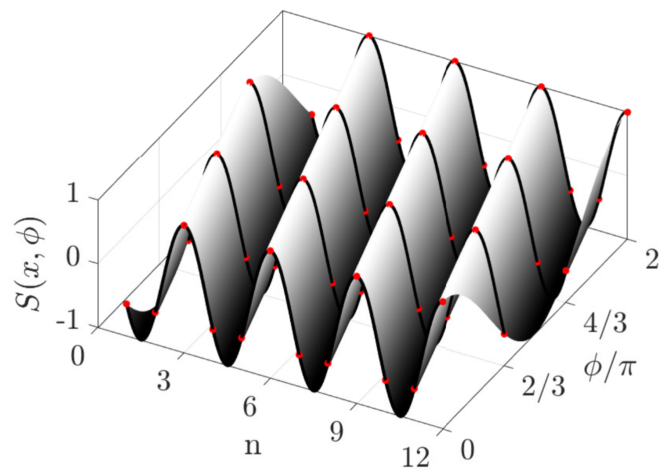

FIGURE 2. 2D SURFACE $S(x,\phi) = cos(2\pi Qx+\phi)$ SAMPLED TO GENERATE SPRING STIFFNESS CONSTANTS. SAMPLED AT RED DOTS ALONG BLACK LINES OF CONSTANT PHASE, ϕ

in the resonators, tuning of the shunt circuit is useful for tuning the resonant frequency of the local resonator and the bandgap [17–20]. Various methods have also been explored to increase the energy harvesting performance of these metamaterials such as using graded patterns in the shunt circuits [21] and physically coupling resonators [22].

Quasiperiodic metamaterials also possess unique topological properties that can be useful in improving both vibration control and energy harvesting. One such property is the existence of additional bandgaps that are spanned by edge modes in finite systems [23, 24]. When quasiperiodic patterns and local resonance are used together, multiple bandgaps appear that can be tuned separately by the parameters of the resonators and the quasiperiodic pattern [25]. The introduction of a phase variable to the quasiperiodic pattern also allows for transition of the edge mode from one boundary to the opposite [26]. These localized edge modes may pave the way for effective energy harvesting as only the few cells with large amplitude vibrations will need energy harvesters embedded in them. However, it is currently unknown what effect electromechanical coupling may have on the band structure and localized vibration modes of a quasiperiodic structure.

In this paper we consider a quasiperiodic chain with local electromechanical resonators. The quasiperiodic pattern is present through a variation in the spring stiffness between cells. The electromechanical resonators are shunted to an external load resistor for harvesting the generated power. First, the governing equations of motion are presented, and the dispersion surfaces are determined analytically for an infinite system. Then the band structure is validated through numerical simulation of a semi-infinite system. Vibration mode shapes are plotted for multiple edge modes, and discussion is given for energy harvesting design consideration. Furthermore, the effects of electromechanical coupling parameters on the band structure and edge modes are observed.

2 SYSTEM DESCRIPTION AND MATHEMATICAL MODELING

A schematic of the quasiperiodic structure under consideration is shown in Fig. 1. The structure consists of S crystals of mass M. Within each cell is imbedded a local electromechanical resonator shunted to an external resistor R. The electromechanical resonator has an effective mass, m_p, effective stiffness, k_p, electromechanical coupling coefficient, θ, and capacitance of the piezoelectric element, C_p. Each cell is connected by springs whose constant k_n is defined by the sampling of the 2D surface $S(x,\phi) = cos(2\pi Qx+\phi)$ at $x = x_n$ (Fig. 2). This surface can be defined by the quasiperiodic parameter, Q, and the phase variable, ϕ. As such, the spring constant is defined as

$$k_n = k_0[1+\alpha cos(2\pi Qn+\phi)] \tag{1}$$

The governing equations of motion for the n$^{\text{th}}$ mass and electromechanical resonator are

$$m\ddot{u}_n + (k_{n-1}+k_n)u_n - k_{n-1}u_{n-1} - k_n u_{n+1} + m_p(\ddot{y}_n+\ddot{u}_n) = 0 \tag{2}$$

$$m_p\ddot{y}_n + k_p y_n - \theta v_n = -m_p\ddot{u}_n \tag{3}$$

$$RC_p\dot{v}_n + v_n + R\theta\dot{y}_n = 0 \tag{4}$$

where $y_n = Y_n - u_n$ is the relative displacement of the n$^{\text{th}}$ resonator relative to the n$^{\text{th}}$ mass.

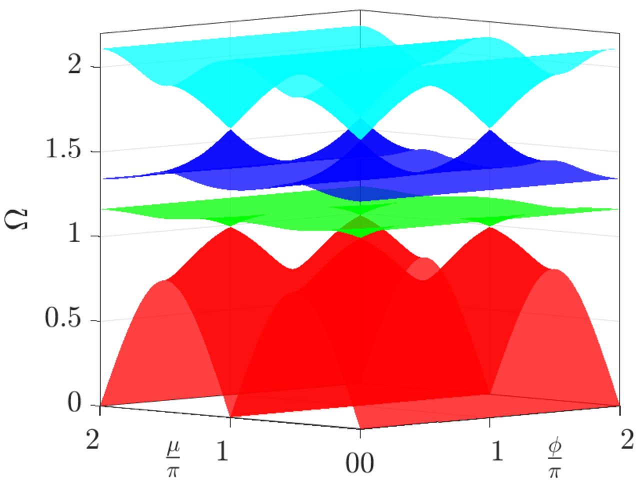

FIGURE 3. DISPERSION SURFACES AS A FUNCTION OF μ AND ϕ FOR $Q = 1/2$ SHOWING FOUR BULK BANDS AND THREE BANDGAPS

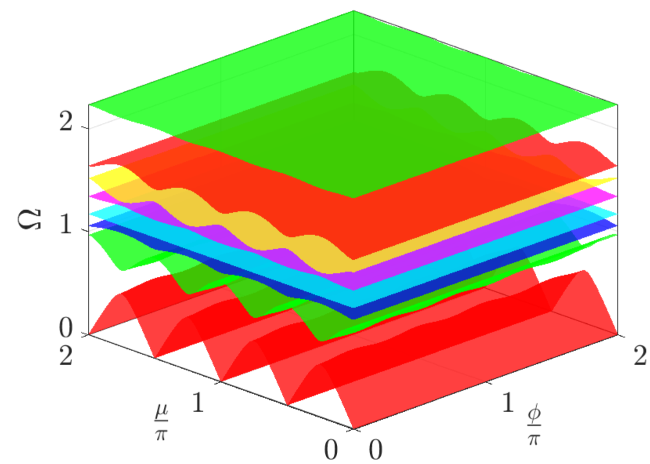

FIGURE 4. DISPERSION SURFACES AS A FUNCTION OF μ AND ϕ FOR $Q = 1/4$ SHOWING EIGHT BULK BANDS AND SEVEN BANDGAPS

Imposing a Bloch periodic solution of

$$u_n = \bar{U}_n e^{j(\mu n - \omega t)} \tag{5}$$

$$y_n = \bar{Y}_n e^{j(\mu n - \omega t)} \tag{6}$$

$$v_n = \bar{V}_n e^{j(\mu n - \omega t)} \tag{7}$$

where μ is the non-dimensional wavenumber, will yield the governing equations

$$(-m\omega^2 + k_{n-1} + k_n)\bar{U}_n - k_{n-1}\bar{U}_{n-1}e^{-j\mu} - k_n\bar{U}_{n+1}e^{j\mu} - m_p\omega^2\bar{Y}_n = 0 \tag{8}$$

$$(-m_p\omega^2 + k_p)\bar{Y}_n - k_p\bar{U}_n - \theta\bar{V}_n = 0 \tag{9}$$

$$\left(-j\omega + \frac{1}{RC_p}\right)\bar{V}_n - j\omega\frac{\theta}{C_p}\bar{Y}_n + j\omega\frac{\theta}{C_p}\bar{U}_n = 0 \tag{10}$$

This system of equations provides an analytical relationship between the frequency and wavenumber which will be used to obtain the dispersion surfaces for an infinite structure. Further analysis of a semi-infinite structure yields the eigenvalues and eigenvectors, giving us the natural frequencies and mode shapes of the structure.

For this study, we will be observing the band structure for a full range of quasiperiodic parameters with a greater focus on $Q = 1/2$ and $1/4$. Although the system is only quasiperiodic for irrational Q values and periodic for rational Q values, the spectrum depends continuously on Q. As such, it can be accurately represented through sampling over rational values of Q [27]. Numerical simulation is used to determine the natural frequencies for a semi-infinite system varying the phase variable, ϕ, as well as the quasiperiodic parameter, Q.

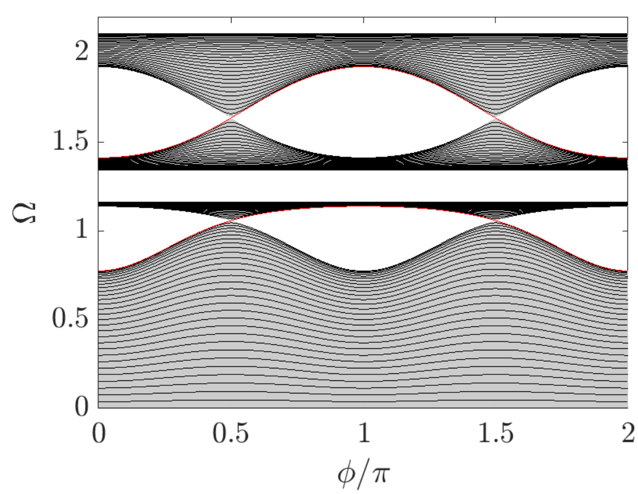

FIGURE 5. NATURAL FREQUENCIES FOR A CHAIN OF $S = 60$ CELLS (BLACK LINES) SUPERIMPOSED ON BULK BANDS (SHADED GREY) WITH VARIATION IN THE PHASE VARIABLE, ϕ and $Q = 1/2$.

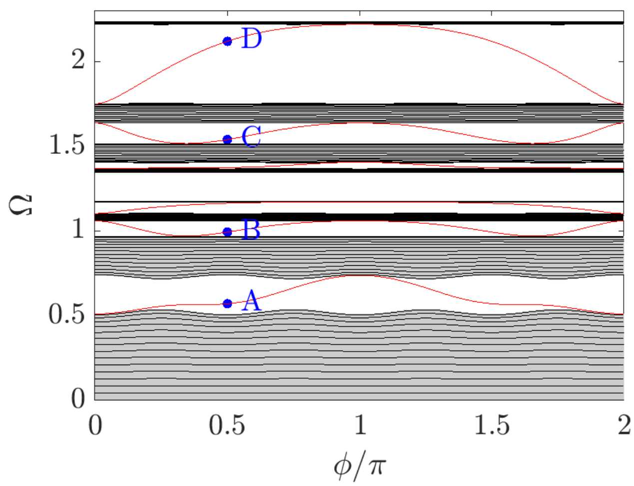

FIGURE 6. NATURAL FREQUENCIES FOR A CHAIN OF $S=60$ CELLS (BLACK LINES) SUPERIMPOSED ON BULK BANDS (SHADED GREY) WITH VARIATION IN THE PHASE VARIABLE, ϕ and $Q=1/4$.

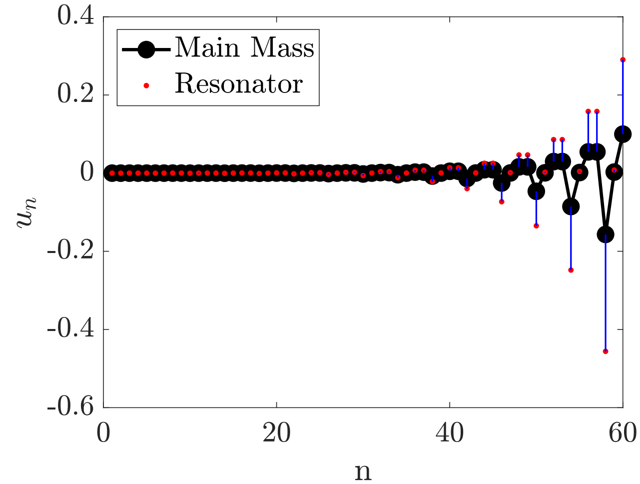

FIGURE 8. LOCALIZED MODE SHAPE FOR A CHAIN OF $S=60$ MASSES (BLACK) AND ELECTROMECHANICAL RESONATORS (RED) WITH $Q=1/4$ CORRESPONDING TO THE NATURAL FREQUENCY at point 'B' in Fig. 6.

3 EFFECT OF ELECTROMECHANICAL RESONATOR ON THE BAND STRUCTURE

Here we will consider a semi-infinite chain of $S=60$ masses and resonators with the following parameters: $m=1$ kg, $k_0=1$ N/m, $m_p=0.2$ kg, $k_p=0.3$ N/m, $R=10$ MΩ, $C_p=13.3$ nF, $\theta=10^{-10}$ N/V. After calculating the roots of the system of equations derived from equations 8-10, we plot the dispersion surfaces for the infinite structure as a function of the non-dimensional wavenumber, μ and the phase variable, ϕ. We can see these surfaces in Figs. 3 and 4 for $Q=1/2$ and $1/4$, respectively.

In each case, the dispersion relations identically mirror those for a structure without electromechanical coupling. For this rea-

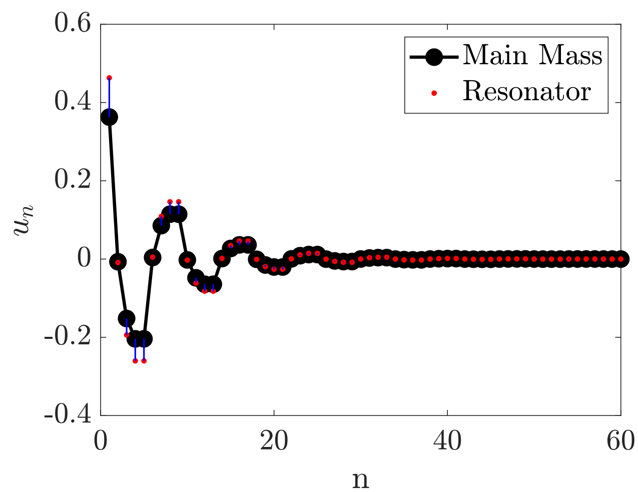

FIGURE 7. LOCALIZED MODE SHAPE FOR A CHAIN OF $S=60$ MASSES (BLACK) AND ELECTROMECHANICAL RESONATORS (RED) WITH $Q=1/4$ CORRESPONDING TO THE NATURAL FREQUENCY at point 'A' in Fig. 6.

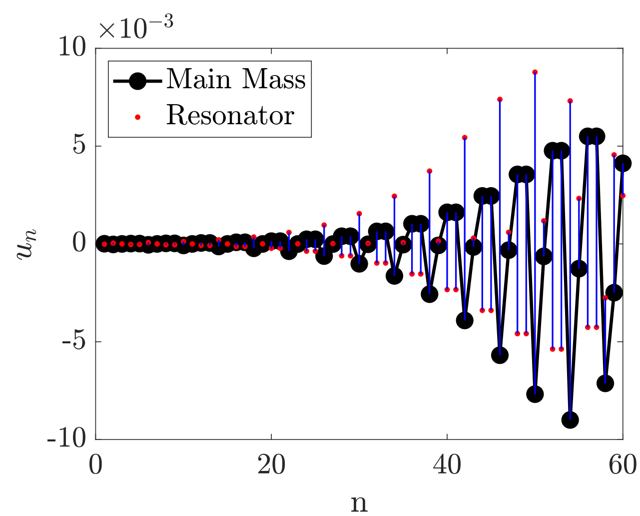

FIGURE 9. LOCALIZED MODE SHAPE FOR A CHAIN OF $S=60$ MASSES (BLACK) AND ELECTROMECHANICAL RESONATORS (RED) WITH $Q=1/4$ CORRESPONDING TO THE NATURAL FREQUENCY at point 'C' in Fig. 6.

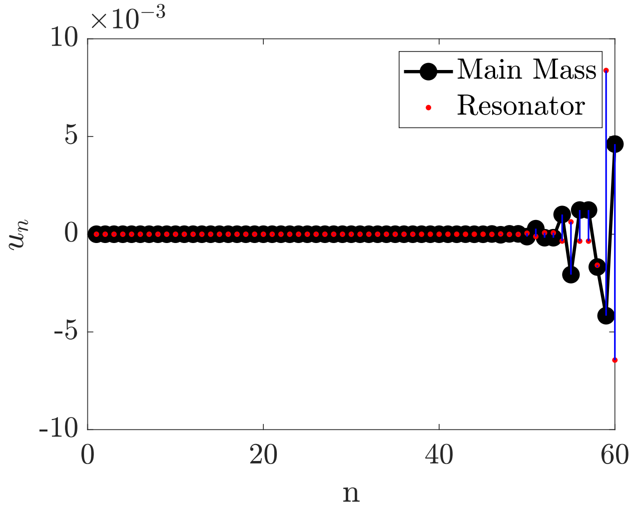

FIGURE 10. LOCALIZED MODE SHAPE FOR A CHAIN OF $S = 60$ MASSES (BLACK) AND ELECTROMECHANICAL RESONATORS (RED) WITH $Q = 1/4$ CORRESPONDING TO THE NATURAL FREQUENCY at point 'D' in Fig. 6.

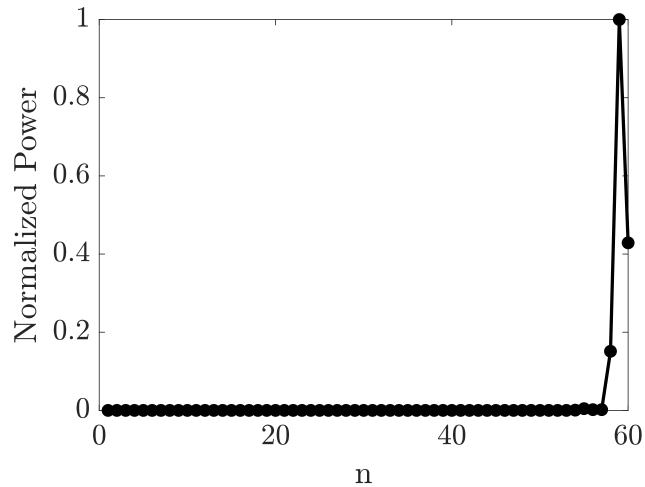

FIGURE 12. RELATIVE POWER HARVESTED FROM A CHAIN OF $S = 60$ CELLS WITH $Q = 1/4$ EXCITED WITHIN THE LAST BANDGAP AT POINT 'D' IN FIG. 6.

son, the dispersion surfaces for a structure without electromechanical coupling are not included here. There is a clearly defined bandgap centered on the resonant frequency of the local resonator splitting the surfaces into two bulk surfaces. Above and below this bandgap, each surface is further split into multiple bulk bands separated by a number of bandgaps determined by the quasiperiodic parameter.

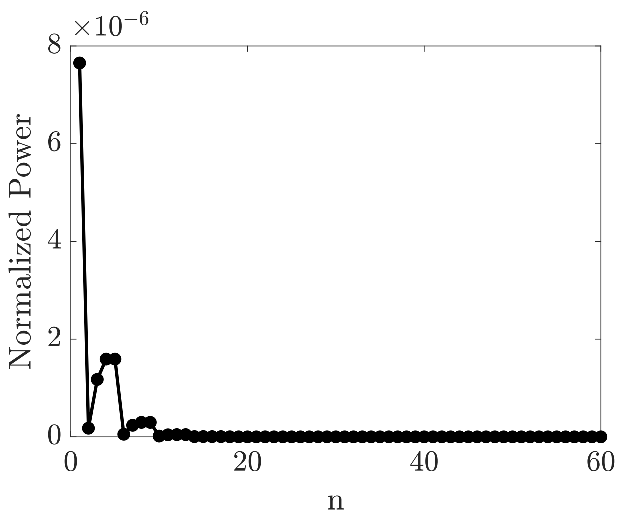

FIGURE 11. RELATIVE POWER HARVESTED FROM A CHAIN OF $S = 60$ CELLS WITH $Q = 1/4$ EXCITED WITHIN THE FIRST BANDGAP AT POINT 'A' IN FIG. 6.

To further validate the analytical results, we plot the natural frequencies for semi-infinite chains in black over the bulk band structure in grey. These are plotted in Figs. 5 and 6 for $Q = 1/2$ and $1/4$, respectively. For the semi-infinite systems, we see not only the bands matching the bulk band, but also additional modes that span the bandgaps (highlighted in red) migrating from one boundary to the opposite as the phase varies. These additional modes often indicate the presence of edge states with localized vibration modes and are a recognizable feature of a finite quasiperiodic structure.

These edge states can be easily observed by plotting the mode shapes of the structure corresponding to natural frequencies within the bandgaps. To demonstrate this, the mode shapes are given for the first and last two bandgaps in the structure with $Q = 1/4$ and $\phi = \pi/2$. These points are labeled A-D in Fig. 6 with corresponding mode shapes plotted in Figs. 7-10, respectively. The masses are shown as black circles with the local resonators included in red. In all cases, the area of large relative vibration is limited to a small portion of the structure, though we do see greater localization in some modes than in other. For example, in the last bandgap, only about eight cells undergo significant vibration while in the second to last bandgap, around thirty cells undergo significant vibration. It is important to note here that the waves are not evanescent or rapidly decaying due to damping. The abrupt decay in wave amplitude is due to the quasiperiodic patterning in the metamaterial rather than electromechanical damping.

In addition, one can observe that in the first two bandgaps, the resonators are moving in phase with the masses while in the last two bandgaps, the resonators are moving out of phase

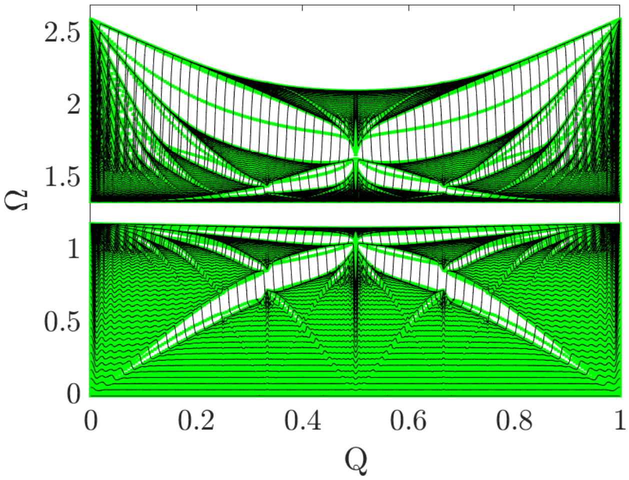

FIGURE 13. SPECTRUM OF NATURAL FREQUENCIES FOR A CHAIN OF $S = 60$ CELLS (BLACK LINES) SUPERIMPOSED OVER A BULK SPECTRUM OF $S = 600$ CELLS (SHADED GREEN) WITH $\theta = 10^{-10}$ N/V, $\phi = 0$ AND VARIATION IN THE QUASIPERIODIC PARAMETER, Q.

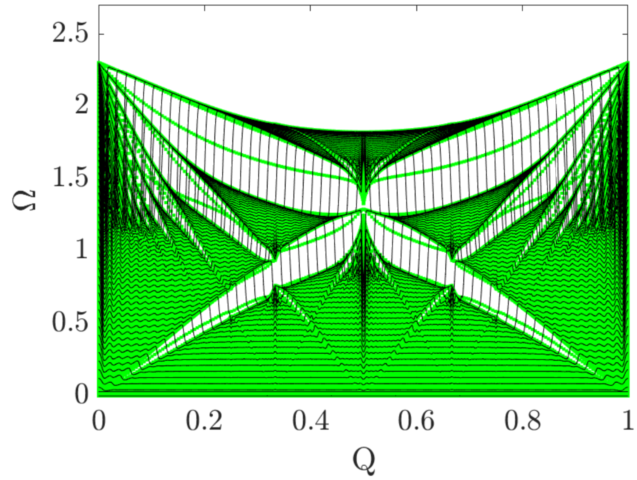

FIGURE 14. SPECTRUM OF NATURAL FREQUENCIES FOR A CHAIN OF $S = 60$ CELLS (BLACK LINES) SUPERIMPOSED OVER A BULK SPECTRUM OF $S = 600$ CELLS (SHADED GREEN) WITH $\theta = 10^{-3}$ N/V, $\phi = 0$ AND VARIATION IN THE QUASIPERIODIC PARAMETER, Q.

with the masses. This shows that for bandgaps occurring below the resonators' natural frequency, the resonators will be in phase with the main masses. Similarly, at frequencies above the resonators' natural frequency, the resonators will be out of phase with the main masses. This can be leveraged to obtain more effective energy harvesting by exciting structures within bandgaps above the resonators' natural frequency. With the resonators moving out of phase of the masses, there will be greater relative displacement leading to greater voltage difference in the piezoelectric material.

To further validate this, the instantaneous power from each cell is plotted from the voltage mode shapes in Figs. 11 and 12 within the first and last bandgaps, respectively. For comparison, both are plotted relative to the maximum power obtained from a cell in either mode. It is shown that the relative power harvested in the last bandgap is multiple orders of magnitude larger than that in the first bandgap. It is also worth noting that in any of the mode shapes, the majority of the power comes from just a handful of cells. In the case of excitation within the last bandgap, 99% of the power is harvested from the last three cells alone.

The band structure is also plotted over the full range of quasiperiodic parameters in Fig. 13 resembling the Hofstadter butterfly commonly found in quantum mechanics. Here the natural frequencies of the semi-infinite system are plotted in black over the bulk bands approximated from $S = 600$ cells in green. From this, we can see that the bandgap produced from the local resonators is topologically trivial as it remains constant with variation in the quasiperiodic parameter. The other bandgaps, however, are topologically nontrivial and depend greatly on the quasiperiodic parameter. We also see again that in the semi-infinite case, there are edge modes spanning each of the topologically non-trivial bandgaps. These results were further tested with variation in the load resistor with resistance values of 10 $\Omega \leq R \leq 10^7$ Ω. The resulting band structures showed no changes due to variation in the load resistance. The results indicate that weak electromechanical coupling has no noticeable effect on the topology of the band structure. This is in good agreement with the results obtained by [16].

To test the case of stronger electromechanical coupling, the band structure was obtained for a range of coupling coefficients of 10^{-10} N/V $\leq \theta \leq 10^{-1}$ N/V. For coupling coefficients $\theta \leq 10^{-5}$ N/V, there is no noticeable change to the band structure. However, as the electromechanical coupling gets stronger, the band structure begins to degrade until the bandgaps collapse on themselves. This can be seen in Fig. 14 for a coupling coefficient of $\theta = 10^{-3}$ N/V. In this case, the topologically trivial bandgap appears to have completely collapsed, and multiple of the other bandgaps have merged together. This indicates that for very strong electromechanical coupling, the band structure may be significantly altered. However, it is also worth noting that in most engineering applications, it is uncommon for the coupling coefficient to exceed the order of 10^{-10} N/V.

4 CONCLUSION

In this paper, we investigated the effect of energy harvesting on the band structure topology and localized vibration modes of locally resonant quasiperiodic metamaterials. The system under consideration was represented by a semi-infinite chain of spring-mass elements with the spring stiffness constants varying in a quasiperiodic pattern. Each cell is connected to an electromechanical local resonator modeled as a spring-mass system and shunted to a load resistor. Analytical dispersion surfaces are given for an infinite system and validated numerically for a semi-infinite system.

The system is shown to have topologically nontrivial bandgaps determined by the quasiperiodic parameter. When the structure is excited within one of these bandgaps, it will produce edge-localized vibration modes. At frequencies above the local resonators' natural frequency, the masses and resonators are out of phase leading to greater energy harvested. Within any bandgap, the majority of energy is harvested from a small percentage of cells in the structure.

The band structure of the semi-infinite system is determined for a full range of phase variables and quasiperiodic parameters. It is also tested for different load resistors and electromechanical coupling values. The results show that quasiperiodic metamaterials with local resonators can be used to harvest energy without changing the topology of the bandgaps for the case of weak electromechanical coupling. However, very strong coupling can cause the performance to degrade as the band structure deforms and bandgaps collapse. Electromechanical coupling also has no negative impact on the ability of a quasiperiodic metamaterial to host localized modes of vibration. When compared to the state of the art periodic energy harvesting metamaterials, the proposed quasiperiodic metamaterial provides greater efficiency in energy harvesting. With vibrations localized to a small portion of the metamaterial, a much smaller percentage of electromechanical energy harvesters is required to harvest a comparable amount of energy.

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