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Optical-acoustic mode coupling in a hierarchical nonlinear acoustic metamaterial

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ABSTRACT

Locally resonant nonlinear acoustic metamaterials can provide low-frequency control and bandwidth extension, but many designs are narrow-band, have limited tunability and robustness, and face low-frequency–thickness trade-offs. To address these limitations, this work investigates wave propagation in a one-dimensional inward configuration hierarchical metamaterial with weakly nonlinearity using an approximate analytical method of Lindstedt–Poincaré and numerical simulations; the analytical model captures the acoustic and first-optical branches, while the second-optical branch is characterized numerically. Spectro-spatial analysis reveals coupling between the acoustic and optical branches, which broadens the effective bandwidth. From the dispersion relation, we observe that chain hardening/softening primarily upshifts/downshifts the acoustic and second-optical branches, whereas resonator nonlinearity mainly tunes the optical branches; the first-optical branch is largely insensitive to chain nonlinearity. Consistent with these shifts, under hardening nonlinearity, the chain generates solitary-like and dispersive packets, whereas with softening, propagation is primarily dispersive. Finally, two passive, time-invariant diode architectures are numerically examined to demonstrate that hierarchy-enabled optical-acoustic mode coupling can produce direction-dependent transmission, with an explicit efficiency-isolation trade-off. These results suggest that hierarchy-enabled optical-acoustic coupling may provide a passive mechanism for broadband wave-control concepts and direction-biased device designs, with potential relevance to energy-transfer and sensing applications.

1. Introduction

Inspired by photonic crystals, phononic and locally resonant acoustic metamaterials are subwavelength structures composed of chains of unit cells with internal resonators. Their behavior arises from two mechanisms: (i) Bragg scattering, which opens bandgaps when the lattice period is comparable to the wavelength [1,2]; and (ii) local resonance, which enables subwavelength control by creating low-frequency stop bands with tuned resonators [3,4]. These materials support applications in acoustic and vibration insulation [5,6], sound absorption [7], cloaking [8], sensing [9,10], and topological insulators [11]. Furthermore, recent studies have leveraged asymmetry to (i) thermally program loss and resonance properties for direction-dependent propagation [12], (ii) achieve switchable bidirectional absorption via interleaved resonator coupling and exceptional point modulation [13], and (iii) realize architected materials with unified strong cross-domain coupling for multifunctionality [14]. Nonetheless, most existing studies adopt a linear assumption, which often yields narrow bandwidths and performance that depends strongly on the attached mass ratio (resonator-to-host).

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To extend the functionality of acoustic metamaterials beyond the linear regime, nonlinearity has been introduced in the metamaterial [15–17]. The nonlinearity can be introduced in the internal resonators [18], in the host chain [19,20], or in both [21]. The source of nonlinearity causes the structure to exhibit distinct spectral consequences. Nonlinearity in the host chain primarily affects short-wavelength behavior; resonator nonlinearity leads to long-wavelength response; while combining nonlinearities in the chain and multiple resonators influences both regions. These nonlinear configurations enable amplitude-dependent transmission [22], solitary and dispersive wave packets [17], and nonreciprocity [23,24]. For instance, nonlinear resonators can broaden stop bands and enable quasi-static bandgap tunability, and strongly nonlinear lattices can support multiple families of traveling breathers [25].

In nonlinear acoustic metamaterials, architecture and damping together govern packet morphology, spectral shifts, and operating range. When damping is present, nonlinear single-resonator [20] and nonlinear diatomic chains [26] modify transient packets: a single-resonator chain shows amplitude reduction, and a diatomic chain shows changes in packet shape and amplitude. In the same context, resonator-only nonlinearity in a spatially antisymmetric Kresling–origami metamaterial yields a dominant linear contour tied to a solitary wave and a separate dispersive pattern [18]. Additionally, weakly nonlinear electromechanical resonators produce wave distortion and a long-wavelength frequency shift [27]; adding a weakly nonlinear chain further enables energy transfer between frequency components. Nonlinear local resonators with RL shunts enhance wave damping, accompanied by distortion [28]. However, most analyses consider a single resonator, which limits the operating range compared with unit cells containing multiple masses. Acoustic metamaterials with multiple masses in parallel configurations have been reported [29]. Although such systems exhibit multiple mode branches in their dispersion relations, the output spectrum remains confined to the same modal branch as the input excitation.

To expand the operating range, multi-resonator unit cells have been explored. Al-Sheyyab and Bukhari analyzed a metamaterial with a parallel pair of local resonators in each unit cell [30]. They observed that the vibro-impact mechanism enables bidirectional energy transfer and mode conversion between low- and high-frequency bands, thereby achieving targeted energy transfer. The use of multi-resonator layouts motivates hierarchical architectures in which resonators are organized across levels [31,32]. Furthermore, resonator arrangements in linear hierarchical metamaterials govern band characteristics: outward and inward hierarchical configurations have been analyzed for mechanical and electromechanical resonators [33,34]. The linear hierarchical configuration increases the number of accessible branches and introduces additional cutoffs; these arrangements also shift band positions and widths in mechanical and electromechanical implementations. Hierarchical layouts also support lightweight structural systems [35]. Compared with linear non-hierarchical counterparts, the hierarchical configuration yields finer spectral control and enables additional routing possibilities within the same footprint.

Despite the advantages of lightweight linear hierarchical acoustic metamaterials, they lack pseudo-bandgaps as observed in the nonlinear metamaterials [31]. In nonlinear metamaterials, frequency and wavenumber shifts generate pseudo-bandgaps that block waves that would otherwise be transmitted by linear media. Zhou et al. showed that a Hann-modulated transient in a locally resonant nonlinear acoustic metamaterial produces such shifts and a pseudo-bandgap [19]. Further, coupling weakly nonlinear chains with electromechanical resonators enables broadband acoustic nonreciprocity, implemented in electromechanical diodes [36]. These frequency-shift approaches provide direction-dependent filtering but do not extend operation beyond the medium- and long-wavelength regimes of the optical mode. Nonlinearity in the metamaterial also enables mode conversion between wave modes [37]. Direct conversion is typically narrow-band, weakly tunable, and inefficient, whereas second-harmonic generation (SHG) in locally resonant systems improves bandwidth and tunability [38]. Interaction between nonlinearity and local resonators establishes phase matching between the initial and converted harmonics, enhancing conversion efficiency; reported SHG routes include coupled translation–rotation [39,40] and nonlinearity-activated intermodal tunneling [41]. Yet these mechanisms operate within or between existing branches, keeping performance branch-dependent motivating architectures that reshape the branch landscape itself.

While nonlinear acoustic metamaterials have been widely studied, most efforts have been focused on dispersion phenomena [42, 43]. Specifically, prior studies have examined solitary and dispersive waves and frequency shifts predominantly within the medium- and long-wavelength regimes of the optical branch [27], resulting in a narrow operating spectrum that limits the robustness and versatility of wave-based devices. Structural asymmetry arising from spatially varying lattice properties in a nonlinear system leads to direction-dependent wave transmission and acoustic nonreciprocity [44]. However, the resulting nonreciprocal response is typically restricted to a single passband. Moreover, lightweight architectures that intentionally couple acoustic and optical modes across wavelength regimes remain unexplored [45,46]. This gap motivates the present inward-configured hierarchical nonlinear metamaterial (HNM), in which nonlinear resonators are arranged to deliberately promote optical–acoustic mode coupling and provide additional propagation channels. The resulting nonlinear hierarchical optical–acoustic mode coupling enables multiple transmission channels, extending beyond the single-channel behavior reported in previous studies. We introduce nonlinearities in both the main chain and the local resonators to investigate their impact across all wavelength regimes. Methodologically, we develop an approximate analytical approach to predict dispersion cutoff frequencies and perform a numerical spectro-spatial analysis to characterize optical–acoustic coupling, frequency shifting, wave distortion, and mode interactions. Collectively, these results demonstrate architecture-enabled inter-branch coupling and bandwidth expansion within a hierarchical, lightweight design.

The rest of the paper is organized as follows. Section 2 introduces the modeling framework and solution methodology. In Section 3, the dispersion relations are investigated both analytically and numerically, followed by the spectro-spatial analysis in Section 4, which focuses on the evolution of spatiotemporal wave packets. The underlying mechanism of optical–acoustic mode coupling is then examined in detail in Section 5. Building on these results, Section 6 demonstrates the design of direction-biased mechanical wave-guiding diodes that integrate two complementary techniques to highlight the key findings. Finally, conclusions and advantages enabled by optical–acoustic coupling are discussed in Section 7.

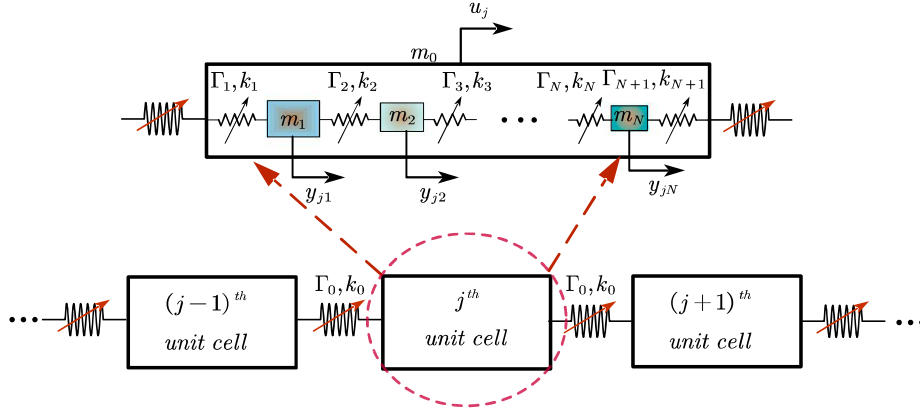


Fig. 1. Chain of hierarchical nonlinear metamaterial.

2. Mathematical model and perturbation analysis

In this section, we develop a discrete chain model for the hierarchical nonlinear metamaterial and carry out a weakly nonlinear analysis using the Lindstedt–Poincaré method. The formulation introduces an inward cascaded chain of local resonators within each unit cell, retaining weakly cubic restoring forces.

2.1. Model description of HNM

An inward configured, nonlinear mass-in-mass hierarchical acoustic metamaterial is shown in Fig. 1. Each unit cell comprises an outer mass m_0 that couples to adjacent cells through linear and nonlinear stiffness elements k_0 and Γ_0 , respectively. The outer mass also hosts a cascade chain of N internal resonators with masses $m_1, m_2, \dots$ and m_N . Every adjacent resonator pair (m_n, m_{n+1}) is connected via linear and nonlinear stiffness elements k_{n+1} and Γ_{n+1} for $n = 1, \dots, N-1$, while resonator m_1 and m_N are connected to outer mass via k_1, Γ_1 and k_{N+1}, Γ_{N+1} , respectively as illustrated in Fig. 1.

Accordingly, the equations of motion for unit cell j and its internal resonators are:

$$m_0 \ddot{u}_j + k_0(2u_j - u_{j-1} - u_{j+1}) + k_1(u_j - y_{j1}) + k_{N+1}(u_j - y_{jN}) + \epsilon \Gamma_0[(u_j - u_{j-1})^3 + (u_j - u_{j+1})^3] + \epsilon \Gamma_1(u_j - y_{j1})^3 + \epsilon \Gamma_{N+1}(u_j - y_{jN})^3 = 0, \quad (1a)$$

$$m_1 \ddot{y}_{j1} + k_1(y_{j1} - u_j) + k_2(y_{j1} - y_{j2}) + \epsilon \Gamma_1(y_{j1} - u_j)^3 + \epsilon \Gamma_2(y_{j1} - y_{j2})^3 = 0, \quad (1b)$$

$$\vdots$$

$$m_n \ddot{y}_{jn} + k_n(y_{jn} - y_{j(n-1)}) + k_{n+1}(y_{jn} - y_{j(n+1)}) + \epsilon \Gamma_n(y_{jn} - y_{j(n-1)})^3 + \epsilon \Gamma_{n+1}(y_{jn} - y_{j(n+1)})^3 = 0, \quad (1c)$$

$$\vdots$$

$$m_N \ddot{y}_{jN} + k_N(y_{jN} - y_{j(N-1)}) + k_{N+1}(y_{jN} - u_j) + \epsilon \Gamma_N(y_{jN} - y_{j(N-1)})^3 + \epsilon \Gamma_{N+1}(y_{jN} - u_j)^3 = 0. \quad (1d)$$

where u_j , and y_{jn} denote the displacements of the j th outer mass and the n th local resonator in it, respectively, and ϵ is a small, dimensionless perturbation (bookkeeping) parameter indicating weak nonlinearity. The equations above describe the j th unit cell with N local resonators arranged in an inward cascaded chain. In the rest of the analysis, we restrict our attention to $N = 2$. This reduction captures the essential nonlinear dynamics, including the first two internal-resonance families, while maintaining algebraic tractability for closed-form developments. Additional resonators ($N > 2$) introduce further optical branches and ancillary stop bands but do not alter the underlying mechanism; accordingly, the governing equations of motion for two internal resonators are (Eq. (1)):

$$m_0 \ddot{u}_j + k_0(2u_j - u_{j-1} - u_{j+1}) + k_1(u_j - y_{j1}) + k_3(u_j - y_{j2}) + \epsilon \Gamma_0[(u_j - u_{j-1})^3 + (u_j - u_{j+1})^3] + \epsilon \Gamma_1(u_j - y_{j1})^3 + \epsilon \Gamma_3(u_j - y_{j2})^3 = 0, \quad (2a)$$

$$m_1 \ddot{y}_{j1} + k_1(y_{j1} - u_j) + k_2(y_{j1} - y_{j2}) + \epsilon \Gamma_1(y_{j1} - u_j)^3 + \epsilon \Gamma_2(y_{j1} - y_{j2})^3 = 0, \quad (2b)$$

$$m_2 \ddot{y}_{j2} + k_2(y_{j2} - y_{j1}) + k_3(y_{j2} - u_j) + \epsilon \Gamma_2(y_{j2} - y_{j1})^3 + \epsilon \Gamma_3(y_{j2} - u_j)^3 = 0. \quad (2c)$$

Furthermore, we introduce the following dimensionless scales and parameters:

$$\alpha_1 = \frac{k_1}{k_0}, \quad \alpha_2 = \frac{k_2}{k_0}, \quad \alpha_3 = \frac{k_3}{k_0}, \quad \omega_0 = \sqrt{\frac{k_0}{m_0}}, \quad \tau = \omega t, \quad \Omega = \frac{\omega}{\omega_0}.$$

In the above definitions, ω denotes the angular frequency of the wave, ω_0 is a characteristic frequency, Ω represents the normalized frequency and m_r is the total mass of the resonators within the j th unit cell,

$$m_r = m_1 + m_2.$$

Furthermore, using $\eta = \frac{m_r}{m_0}$ and $\beta_1 = \frac{m_1}{m_r}$, the masses for the two-resonator case are

$$m_1 = \eta \beta_1 m_0, \quad m_2 = \eta(1 - \beta_1) m_0.$$

With these nondimensional scales and parameters, Eq. (2) is simplified as

$$\begin{aligned} \Omega^2 \frac{\partial^2 u_j}{\partial \tau^2} + (2u_j - u_{j-1} - u_{j+1}) + \epsilon \gamma_0 [(u_j - u_{j-1})^3 + (u_j - u_{j+1})^3] + \alpha_1 (u_j - y_{j1}) \\ + \alpha_3 (u_j - y_{j2}) + \epsilon \gamma_1 (u_j - y_{j1})^3 + \epsilon \gamma_3 (u_j - y_{j2})^3 = 0, \end{aligned} \quad (3a)$$

$$\eta \beta_1 \Omega^2 \frac{\partial^2 y_{j1}}{\partial \tau^2} + \alpha_1 (y_{j1} - u_j) + \alpha_2 (y_{j1} - y_{j2}) + \epsilon \gamma_1 (y_{j1} - u_j)^3 + \epsilon \gamma_2 (y_{j1} - y_{j2})^3 = 0, \quad (3b)$$

$$\begin{aligned} \eta(1 - \beta_1) \Omega^2 \frac{\partial^2 y_{j2}}{\partial \tau^2} + \alpha_2 (y_{j2} - y_{j1}) + \alpha_3 (y_{j2} - u_j) + \epsilon \gamma_2 (y_{j2} - y_{j1})^3 \\ + \epsilon \gamma_3 (y_{j2} - u_j)^3 = 0. \end{aligned} \quad (3c)$$

where $\gamma_0 = \frac{F_0}{k_0}$, $\gamma_1 = \frac{F_1}{k_0}$, $\gamma_2 = \frac{F_2}{k_0}$, $\gamma_3 = \frac{F_3}{k_0}$. To investigate the dynamics of the proposed hierarchical nonlinear metamaterial, we adopt a hybrid analytical-numerical method. First, a Lindstedt–Poincaré (LP) expansion is applied to derive an amplitude-dependent wavenumber–frequency relations and cutoff trends for the infinite periodic chain. Next, we simulate a finite chain with perfectly matched layers (PMLs) excited by a Hann wave packet, and extract its dispersive content via spectro-spatial analysis using spectrograms and 2D Fourier transforms. This combined approach both validates and extends the LP predictions, especially where higher optical content emerges, thereby clarifying the conditions for optic–acoustic coupling. We proceed with the analytical formulation based on LP.

2.2. Lindstedt–Poincaré analysis

To obtain an analytical, amplitude-dependent dispersion relation, we apply the Lindstedt–Poincaré method to a single unit cell embedded in an infinitely long version of the proposed HNM [47]. Accordingly, we expand Ω , u_j , y_{j1} , and y_{j2} as

$$\Omega = \Omega_0 + \epsilon \Omega_1, \quad (4a)$$

$$u_j = u_j^0 + \epsilon u_j^1, \quad (4b)$$

$$y_{j1} = y_{j1}^0 + \epsilon y_{j1}^1, \quad (4c)$$

$$y_{j2} = y_{j2}^0 + \epsilon y_{j2}^1. \quad (4d)$$

Substituting Eq. (4) into nondimensional equations of motion (Eq. (3)) and collecting different order of ϵ yields

ϵ^0 :

$$\begin{aligned} \Omega_0^2 \frac{\partial^2 u_j^0}{\partial \tau^2} + (2u_j^0 - u_{j-1}^0 - u_{j+1}^0) + \alpha_1 (u_j^0 - y_{j1}^0) + \alpha_3 (u_j^0 - y_{j2}^0) = 0, \\ \eta \beta_1 \Omega_0^2 \frac{\partial^2 y_{j1}^0}{\partial \tau^2} + \alpha_1 (y_{j1}^0 - u_j^0) + \alpha_2 (y_{j1}^0 - y_{j2}^0) = 0, \\ \eta(1 - \beta_1) \Omega_0^2 \frac{\partial^2 y_{j2}^0}{\partial \tau^2} + \alpha_2 (y_{j2}^0 - y_{j1}^0) + \alpha_3 (y_{j2}^0 - u_j^0) = 0. \end{aligned} \quad (5)$$

ϵ^1 :

$$\begin{aligned} \Omega_0^2 \frac{\partial^2 u_j^1}{\partial \tau^2} + (2u_j^1 - u_{j-1}^1 - u_{j+1}^1) + \alpha_1 (u_j^1 - y_{j1}^1) + \alpha_3 (u_j^1 - y_{j2}^1) = -2\Omega_0 \Omega_1 \frac{\partial^2 u_j^0}{\partial \tau^2} \\ - \gamma_0 [(u_j^0 - u_{j-1}^0)^3 + (u_j^0 - u_{j+1}^0)^3] - \gamma_1 (u_j^0 - y_{j1}^0)^3 - \gamma_3 (u_j^0 - y_{j2}^0)^3, \\ \eta \beta_1 \Omega_0^2 \frac{\partial^2 y_{j1}^1}{\partial \tau^2} + \alpha_1 (y_{j1}^1 - u_j^1) + \alpha_2 (y_{j1}^1 - y_{j2}^1) = -2\eta \beta_1 \Omega_0 \Omega_1 \frac{\partial^2 y_{j1}^0}{\partial \tau^2} + \gamma_1 (y_{j1}^0 - u_j^0)^3 \\ - \gamma_2 (y_{j1}^0 - y_{j2}^0)^3, \\ \eta(1 - \beta_1) \Omega_0^2 \frac{\partial^2 y_{j2}^1}{\partial \tau^2} + \alpha_2 (y_{j2}^1 - y_{j1}^1) + \alpha_3 (y_{j2}^1 - u_j^1) = -2\eta(1 - \beta_1) \Omega_0 \Omega_1 \frac{\partial^2 y_{j2}^0}{\partial \tau^2} \\ - \gamma_2 (y_{j2}^0 - y_{j1}^0)^3 - \gamma_3 (y_{j2}^0 - u_j^0)^3. \end{aligned} \quad (6)$$

Note that the ϵ^0 equation represents the linearized system. Hence, to extract the linear dispersion, we assume the plane wave harmonic solution in the form of [19]:

$$\begin{cases} u_j^0 = \frac{A_0}{2} e^{i(j\kappa + \tau)} + \text{cc}, \\ y_{j1}^0 = \frac{B_0}{2} e^{i(j\kappa + \tau)} + \text{cc}, \\ y_{j2}^0 = \frac{C_0}{2} e^{i(j\kappa + \tau)} + \text{cc}. \end{cases} \quad (7)$$

where $i = \sqrt{-1}$, κ is the dimensionless wavenumber and cc represents the complex conjugate. We note that B_0 and C_0 are determined from the linear system in terms of A_0 ; their explicit formulas are listed in Appendix. Substituting the ansatz in Eq. (7) into the $\mathcal{O}(\epsilon^0)$ system (Eq. (5)) and then eliminating A_0 , B_0 , and C_0 yields the linear-chain dispersion relation:

$$\begin{aligned} & \alpha_3(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - \alpha_3\beta_1\eta\Omega_0^2) + [\alpha_1 + \alpha_3 - \eta\Omega_0^2(1 - \beta_1)] \\ & \times [\alpha_1^2 - (\alpha_1 + \alpha_2 - \beta_1\eta\Omega_0^2)(\alpha_1 + \alpha_3 - \Omega_0^2 + 2(1 - \cos(\kappa)))] + \\ & \alpha_2[2\alpha_2 + \alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_2\Omega_0^2 - 2\alpha_2\cos(\kappa)] = 0. \end{aligned} \quad (8)$$

Next, we substitute the assumed plane-wave form of the solution (Eq. (7)) into the right-hand side of the $\mathcal{O}(\epsilon^1)$ system (Eq. (6)). Eliminating y_{j1}^1 and y_{j2}^1 in favor of the main-chain displacement u_j^1 yields a single equation of motion for u_j^1 ,

$$\begin{aligned} & [QP(\Omega_0^2 \frac{\partial^2}{\partial \tau^2} + \alpha_1 + \alpha_3) - Q\alpha_3^2 - (P\alpha_1 + \alpha_2\alpha_3)^2]u_j^1 + QP[2u_j^1 - u_{j-1}^1 - u_{j+1}^1] \\ & = (P\alpha_1 + \alpha_2\alpha_3)(PR_2 + \alpha_2R_3) + Q(PR_1 + \alpha_3R_3). \end{aligned} \quad (9)$$

where the auxiliary quantities R_1 , R_2 , R_3 , P , and Q are defined in Appendix. For clarity, we note that P and Q include acceleration terms (i.e., they contain factors proportional to $\partial^2/\partial \tau^2$). While the terms R_1 , R_2 , and R_3 collect the cubic nonlinear contributions evaluated using the zeroth-order fields from Eq. (6).

Furthermore, expanding the right-hand side of Eq. (9) and enforcing the elimination of secular (resonant) forcing yields the first-order dimensionless frequency correction Ω_1 . This correction depends on the zeroth-order dimensionless frequency and the wavenumber, i.e., $\Omega_1 = \Omega_1(\Omega_0, \kappa)$. The explicit expressions are provided in Appendix. Consequently, the weak-perturbation, amplitude-corrected dispersion relation for the nonlinear chain is given by

$$\Omega = \Omega_0 + \epsilon\Omega_1, \quad (10)$$

where the branch (acoustic or optical) is identified by the corresponding eigenvector at $\mathcal{O}(\epsilon^0)$.

The Lindstedt–Poincaré result in Eq. (10) gives the amplitude-corrected dispersion for the infinite chain. To assess its validity at finite amplitudes and to capture phenomena beyond the perturbation regime (e.g., branch mixing), we now reconstruct the dispersion numerically via transient simulations of a long finite chain with Hann-packet excitation, perfectly matched layers (PML) terminations, and 2D-FFT analysis.

3. Dispersion relations analysis

To validate the analytical dispersion relations, we compare them with dispersion curves obtained numerically. A finite chain of 1500 unit cells is excited by a Hann-windowed transient wave packet, and the dynamics is obtained numerically using MATLAB's built-in solver `ode45`. A two-dimensional Fast Fourier Transform (2D-FFT) is then applied to the spatiotemporal amplitude response to reconstruct the dispersion relation in the frequency–wavenumber domain. The dispersion curves are identified by tracking the spectral-amplitude peaks corresponding to dominant frequencies and wavenumbers. Unless otherwise stated, the normalized parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\beta_1 = 0.75$, $\eta = 1$, and $\omega_0 = 10^3$ rad/s.

Furthermore, the initial conditions for the numerical simulations are

$$\begin{cases} u_j(0) = \frac{A_0}{2} \left(H(j-1) - H\left(j-1 - \frac{2\pi N_{cy}}{\kappa}\right) \right) \left(1 - \cos\left(\frac{j\kappa}{N_{cy}}\right) \right) \sin(j\kappa), \\ \dot{u}_j(0) = \frac{A_0}{2} \left(H(j-1) - H\left(j-1 - \frac{2\pi N_{cy}}{\kappa}\right) \right) U_j, \\ y_{j1}(0) = F u_j(0), \\ \dot{y}_{j1}(0) = F \dot{u}_j(0), \\ y_{j2}(0) = J (\alpha_3 u_j(0) + \alpha_2 y_{j1}(0)), \\ \dot{y}_{j2}(0) = J (\alpha_3 \dot{u}_j(0) + \alpha_2 \dot{y}_{j1}(0)). \end{cases} \quad (11)$$

where $H(\cdot)$ is the Heaviside step function, A_0 is the wave-packet amplitude, and $N_{cy} = 7$ is the number of carrier cycles in the Hann-modulated spatial window. The quantity U_j specifies the localized initial velocity profile used to input energy and set the propagation direction, and is given by

$$U_j = \frac{\omega}{N_{cy}} \sin\left(\frac{j\kappa}{N_{cy}}\right) \sin(j\kappa) - \omega \left(1 - \cos\left(\frac{j\kappa}{N_{cy}}\right) \right) \cos(j\kappa). \quad (12)$$

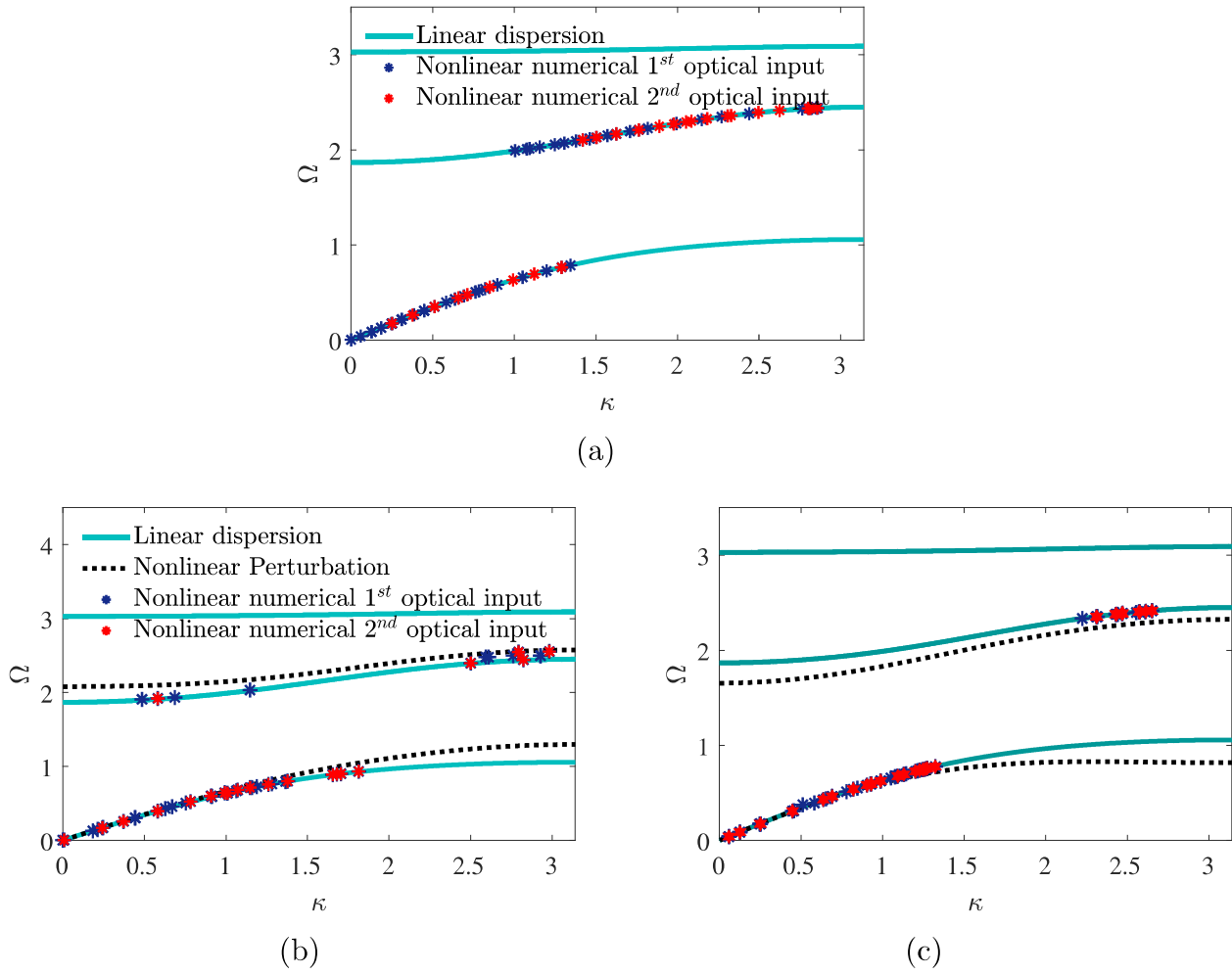


Fig. 2. Comparison of the analytical dispersion relation with numerical results at $\beta_1 = 0.75$: (a) linear dispersion, (b) nonlinear hardening and (c) nonlinear softening.

In the above, F and J are coupling coefficients that link the resonator amplitudes to the chain amplitude (i.e., they relate B_0 and C_0 to A_0) and are obtained from the linear system solution. F and J are functions of the dimensionless stiffnesses, mass ratios, and Ω_0 and are given by

$$\begin{cases} F = \frac{\alpha_2 \alpha_3 + \alpha_1 (\alpha_2 + \alpha_3 + \eta (\beta_1 - 1) \omega_0^2)}{\alpha_2 (\alpha_3 - \eta \omega_0^2) - \eta \beta_1 \omega_0^2 (\alpha_3 + \eta (\beta_1 - 1) \omega_0^2) + \alpha_1 (\alpha_2 + \alpha_3 + \eta (\beta_1 - 1) \omega_0^2)}, \\ J = \frac{1}{\alpha_2 + \alpha_3 + \eta (\beta_1 - 1) \omega_0^2}. \end{cases} \quad (13)$$

To reduce wave reflections, perfectly matched layers (PMLs) are implemented at both ends of the chain, with the damping profile [47]:

$$C(j) = C_{\max} \left(\frac{s}{N_{\text{pml}}} \right)^{d_p}. \quad (14)$$

where $C_{\max} = 2 \times 10^4$ N s/m, $d_p = 3$, and $N_{\text{pml}} = 5$. Here $s = 1, \dots, N_{\text{pml}}$ denotes the local index inside each PML region, counted from the start of the PML toward the interior of the chain (the same profile is mirrored at both ends).

For the linear chain we set $\epsilon \gamma_0 = \epsilon \gamma_i = 0$. For the nonlinear cases we use $\epsilon \gamma_0 = \epsilon \gamma_i = 0.06$ (hardening) and $\epsilon \gamma_0 = \epsilon \gamma_i = -0.06$ (softening), with $i \in \{1, 2, 3\}$. The input wave packet has a normalized amplitude (A_0) of 1. The total simulation time (dimensionless) is $\tau_{\text{end}} = 3000$ for acoustic-mode excitations and $\tau_{\text{end}} = 1500$ for optical-mode excitations.

3.1. Validation of the analytical dispersion relations

In this section, we compare the analytical dispersion relations obtained via the Lindstedt–Poincaré method with numerical dispersion reconstructed from the 2D-FFT of the wave-packet response. The comparison is shown in Fig. 2 for three cases of

linear, hardening, and softening nonlinearity. In all three cases, linear, hardening, and softening, we identify one acoustic branch at low frequencies and two optical branches at higher frequencies. The linear case in Fig. 2(a) provides the baseline for assessing nonlinear shifts, while Fig. 2(b) and (c) show the expected upward and downward shifts for hardening and softening nonlinearity, respectively, relative to the linear curves. From Fig. 2, we observe close agreement between the first-order analytical perturbation and numerical simulations, which validates the analytical approach over the acoustic and first optical branches. Across these panels, the analytical method accurately captures the acoustic and first-optical passbands but does not fully resolve the second optical branch, which shifts significantly beyond the expected frequency range. Consequently, the third passband is not fully visible within the plotted window, since nonlinear effects push it outside the spectro-spatial range. Similar limitations have also been observed for the method of multiple-scales, which loses accuracy when predicting higher-frequency mode branches, especially for nonlinearities above 0.015 [29]. While these approximate analytical approaches can reliably capture lower modes, the upper optical branches become increasingly difficult to resolve. These results demonstrate that, analogous to the method of multiple scales, the Lindstedt–Poincaré method exhibits limited accuracy in capturing higher-order modes of hierarchical nonlinear structures, including the second optical branch. A detailed discussion about the analytical method follows in the next subsection.

Numerical simulations show that excitation near the second optical branch (third passband) produces measurable response on the first optical and acoustic branches, indicating frequency down-conversion as shown in Fig. 2. Prior studies have reported frequency down-conversion only in nonlinear acoustic metamaterials [37]; however, the present metamaterial configuration demonstrates that a linear chain can also facilitate down-conversion of higher-order modes. Because the second optical mode has a relatively narrow bandwidth, the frequency range over which this behavior observed is limited.

We emphasize that, in Fig. 2, near the cutoff frequencies, the numerical method struggles to capture the dominant components because of mode conversion and frequency shifting. Consequently, the long-wavelength components of the optical modes convert into the acoustic mode, while the short-wavelength components of the acoustic mode shift upward in frequency into the first optical mode. In the hardening case (Fig. 2(b)), this interaction gives rise to a pseudo-bandgap in the mid-frequency range of the first optical mode. In the softening case (Fig. 2(c)), the dominant content of the first optical mode is predominantly shifted toward the short-wavelength region. This shift produces a pseudo-bandgap that suppresses wave propagation over the long- and medium-wavelength ranges.

These comparisons confirm the accuracy of the first-order analytical dispersion across the acoustic and first optical branches. The numerical results show that the second optical mode is redistributed into the first optical and acoustic modes due to nonlinear hierarchical interactions. Analytical perturbation approaches are effective for capturing leading-order behavior and providing physical insight into dispersion characteristics. In contrast, higher-order effects such as detailed frequency shifts and complex mode interactions, are more accurately resolved through numerical analysis.

3.2. Influence of chain and resonator nonlinearity on dispersion

Having validated the analytical dispersion relation against numerical simulations, we use this relation to assess the effects of chain and resonator nonlinearities on the dispersion relations.

The dispersion curves for combinations of hardening and softening nonlinearities in the chain and resonators are shown in Fig. 3. In the chain-only case (Fig. 3(a)), regardless of the chain nonlinearity type, the short-wavelength regions of the acoustic and second optical modes are substantially affected, whereas the first optical mode remains essentially unchanged. Consistent with earlier observations, chain hardening shifts these branches upward and chain softening shifts them downward relative to the linear baseline.

Next, to analyze the effect of resonator nonlinearity on the dispersion curves, we first examine the individual effects of the connecting stiffnesses of the local resonators. For nonlinearity in the first connecting stiffness, primarily the first optical mode is affected (Fig. 3(b)). Similar to the observations for chain nonlinearity, hardening shifts the first optical mode upward, while softening shifts it downward. Fig. 3(c) shows the effects of hardening and softening in the second connecting stiffness; in this case, the most pronounced changes occur in the third passband, consistent with prior observations [29] that higher-order branches of the nonlinear dispersion shift. Furthermore, for hardening nonlinearity, the second optical branch shifts upward, whereas for softening, it shifts to lower frequencies in the long-wavelength region.

Finally, the effects of nonlinearity in the third connecting stiffness are shown in Fig. 3(d). Here, the nonlinearity primarily impacts the second optical mode: under softening, the second optical mode shifts downward and approaches the first optical mode, indicating mode conversion, whereas under hardening it shifts slightly upward. Thus, Fig. 3(b)–(d) indicate that resonator nonlinearity, in isolation, is an effective means to tune the optical modes without significantly altering the acoustic behavior.

Finally, we examine the combined effect of chain and resonator nonlinearities on the dispersion relations. Fig. 3(e) shows the impact when both sources are active. As established earlier, chain nonlinearity primarily affects the acoustic branch, whereas resonator nonlinearity tunes the optical modes localized to the resonators. Consequently, their combination influences all branches, as seen in Fig. 3(e). Consistent with prior observations, hardening shifts the modes upward in frequency, while softening shifts them downward in frequency, including corresponding shifts of the bandgap edges.

Since Fig. 3(c) shows that the second optical mode is particularly sensitive to the second resonator's nonlinearity, we consider a case where the chain and the first and third resonators are nonlinear while the second resonator remains linear. The resulting dispersion is shown in Fig. 3(f). As expected, with the second resonator kept linear, the second optical branch exhibits reduced sensitivity compared with Fig. 3(e). In combination, chain and resonator nonlinearities still allow tuning across all branches.

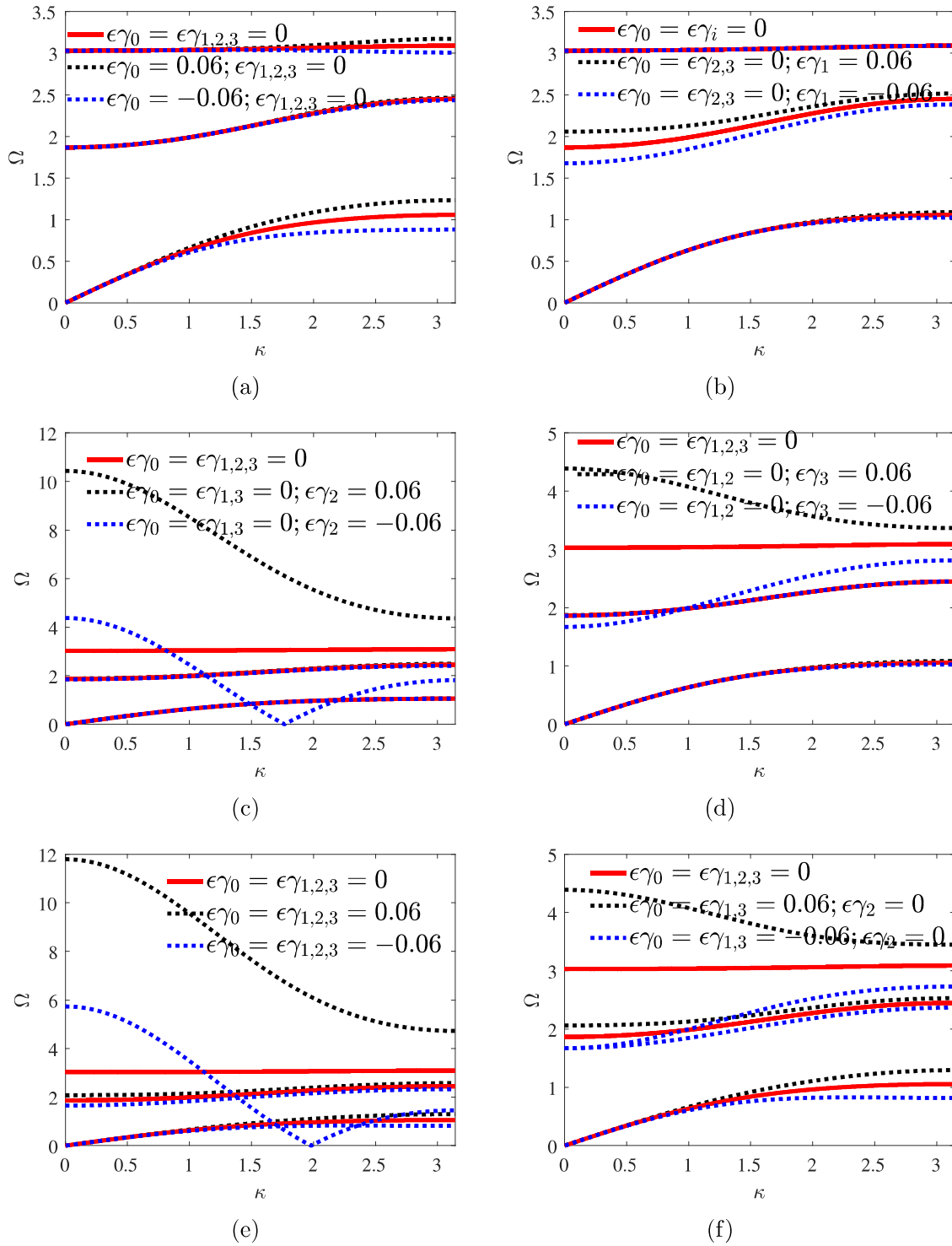


Fig. 3. Dispersion predicated by analytical method at $\beta_1 = 0.75$: (a) chain stiffness hardening and softening, (b) resonator first stiffness hardening and softening, (c) resonator second stiffness hardening and softening, (d) resonator third stiffness hardening and softening, (e) resonators stiffness hardening and softening and (f) chain and resonators stiffness hardening and softening.

4. Spectro-spatial wave packet analysis

Building on the trends of the dispersion curves in Section 3, we now examine finite-duration wave-packet propagation using a spectro-spatial framework. While linear wave propagation can preserve the wave shape, nonlinear propagation can introduce waveform distortion and solitons [29]. Moreover, dispersion relations alone do not provide a decomposition of spatial signals into frequency and wavenumber components. Therefore, in this section we use spectro-spatial analysis to investigate wave propagation

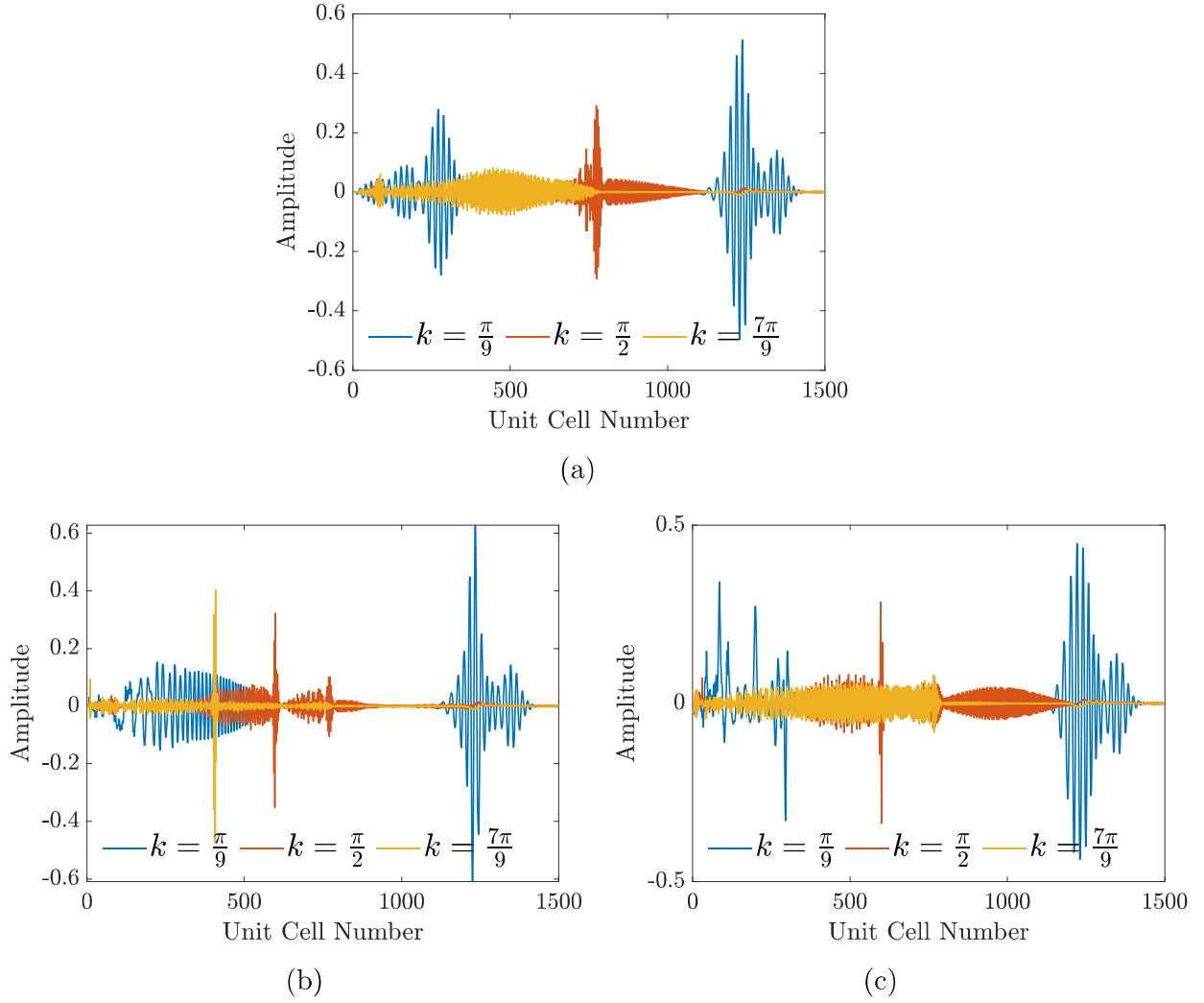


Fig. 4. Spatial profile at the end of simulation when an acoustic wave packet is injected. (a) linear chain, (b) nonlinear hardening, and (c) nonlinear softening.

in the proposed hierarchical metamaterial with the combined nonlinearity of the chain and the local resonators. Spatial profiles are used to examine how this combined nonlinearity alters the shape of the excitation signal as it propagates.

4.1. Spatial profile of transient wave packet

We first examine the normalized spatial profiles to assess waveform shape changes under the combined nonlinearity. As noted earlier in Section 3, in the presence of nonlinearity, a second-optical-mode input down-converts to the first optical-branch at long wavelengths and to the acoustic branch at short wavelengths (Fig. 2(b) and (c)), although the band-width of the second optical branch is relatively narrow. Therefore, in this analysis we focus on excitation of the first optical and acoustic branches.

Fig. 4 shows the normalized spatial behavior (amplitude) of the wave packet propagating in the hierarchical metamaterial, excited by the branch of the acoustic mode at one end of the chain. We observe that excitation using the long-wavelength ($k = \frac{\pi}{9}$) wave packet produces two localized wave packets of varying peak amplitudes traveling in the same direction. The two localized packets appear at different spatial locations at the end of the simulation, consistent with components propagating at different group velocities. This behavior reflects the excitation of two distinct modes in the spectrum (acoustic and optical). As shown in Fig. 4(a), although the shapes of the medium- and short-wavelength wave packets are stretched, the propagation of these wave packets remains undistorted apart from dispersion-induced stretching; this kind of distortion is reported as dispersion-related distortion [19]. Furthermore, the amplitude of the output wave packets from the long-wavelength excitation is smaller than the amplitude of the input wave packets, and this observation holds for the nonlinear chain as well.

However, in the nonlinear chain, input wave packets of long-, medium-, and short-wavelengths experience shape distortion, as shown in Fig. 4(b) and (c). From these figures, we observe that, irrespective of the nonlinearity type, the long-wavelength packet exhibits minimal distortion. In contrast, the medium- and short-wavelength packets become highly dispersive under hardening nonlinearity (Fig. 4(b)), as identified by spread-out, low-amplitude packets; additionally, a localized high-amplitude solitary wave

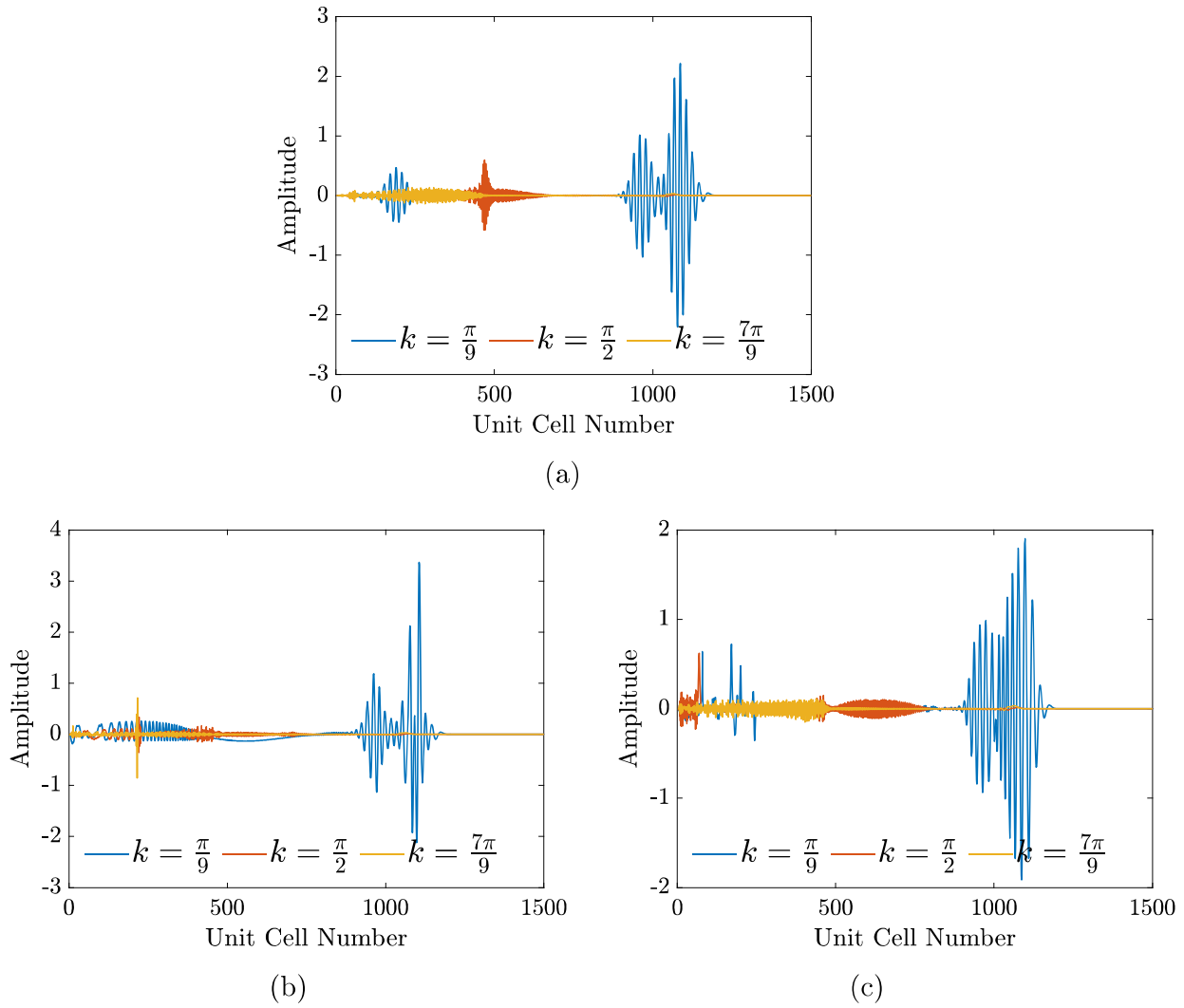


Fig. 5. Spatial profile at the end of simulation when first optical wave packet is injected. (a) linear chain, (b) nonlinear hardening, and (c) nonlinear softening.

is present. Under softening nonlinearity (Fig. 4(c)), similar dispersive behavior is observed in the medium-wavelength regime, but without the formation of solitary waves.

Next, Fig. 5 presents the spatial profiles for excitation of the first optical branch. In the linear chain, as shown in Fig. 5(a), the wave packets exhibit purely dispersive behavior for the medium- and short-wavelength excitations. However, under hardening nonlinearity the wave packets become distorted and exhibit localization; at medium and short wavelengths, the profiles combine dispersive packets with localized solitary waves, as shown in Fig. 5(b). In contrast, the softening nonlinearity (Fig. 5(c)) leads to greater spatial spreading of the wave packets.

These spatial profiles show that chain and resonator nonlinearities significantly modify the propagation of wave packets in the proposed hierarchical metamaterial with cascaded resonators. With hardening nonlinearity, wave packets become increasingly distorted and localized, and both dispersive and solitary features emerge. Conversely, softening nonlinearity enhances dispersion and suppresses solitary-wave formation, supporting smoother propagation where reducing wave intensity is desirable.

4.2. Spectro-spatial evolution and nonlinear mode redistribution

The normalized spatial profiles in the previous subsection describe the influence of chain and resonator nonlinearities on waveform distortion in physical space, but they do not reveal the corresponding energy redistribution over wavenumber. To quantify this redistribution and systematically track mode interactions, we examine the evolution of the wavenumber content through a spatial spectrogram. In this analysis, the chain is excited at the first optical mode using input wavenumbers $\kappa = \frac{\pi}{2}$ and $\frac{7\pi}{9}$. The short-term Fourier transform (STFT) is then applied to compute the spectrogram [48], with the Hann-window length chosen to match the wavelength of the initial burst so that distortion can be compared directly against the input signal.

The spatial spectrograms for the linear chain at $\kappa = \frac{\pi}{2}$ and $\kappa = \frac{7\pi}{9}$ are shown in Fig. 6(a) and (b), respectively. In both cases, the wave packets remain confined within the initial burst window of the spectrogram, marked by the white dashed lines. We observe

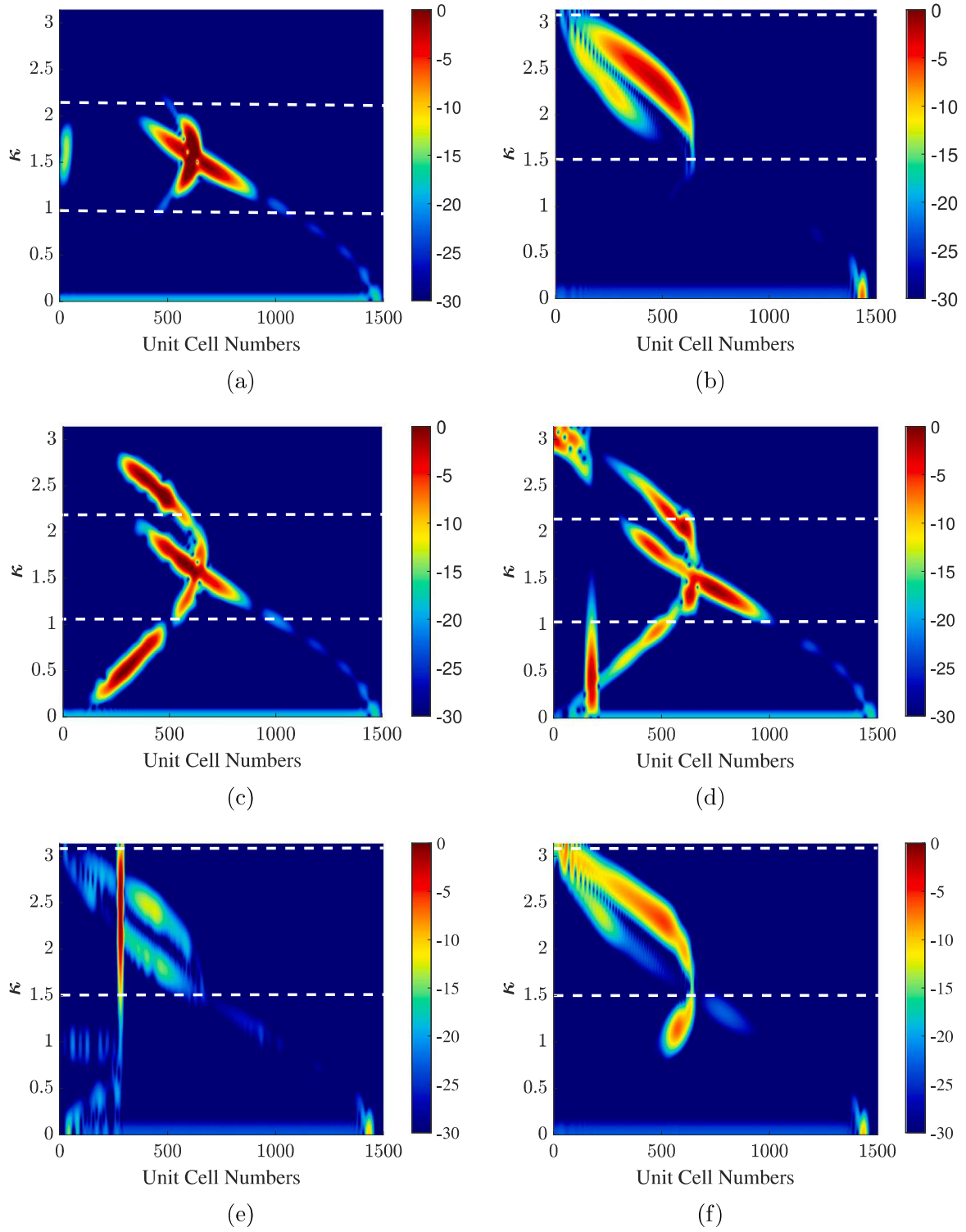


Fig. 6. Spatial spectrogram of the wave packet for the first optical mode excitations at $\beta_1 = 0.75$: (a) $\kappa = \frac{\pi}{2}$, and $\epsilon\gamma_0 = \epsilon\gamma_{1,2,3} = 0$, (b) $\kappa = \frac{7\pi}{9}$, and $\epsilon\gamma_0 = \epsilon\gamma_{1,2,3} = 0$, (c) $\kappa = \frac{\pi}{2}$, and $\epsilon\gamma_0 = \epsilon\gamma_{1,2,3} = 0.06$, (d) $\kappa = \frac{\pi}{2}$, and $\epsilon\gamma_0 = \epsilon\gamma_{1,2,3} = -0.06$, (e) $\kappa = \frac{7\pi}{9}$, and $\epsilon\gamma_0 = \epsilon\gamma_{1,2,3} = 0.06$ and (f) $\kappa = \frac{7\pi}{9}$, and $\epsilon\gamma_0 = \epsilon\gamma_{1,2,3} = -0.06$.

that, although the wavenumber content exhibits mild dispersion-induced spreading, it remains within the initial window. A weak acoustic component is also visible outside the input mode branch. It is also noted that, in the spectrogram of Fig. 6, dispersive wave packets appear as horizontally stretched or slanted features, reflecting their broadened wavenumber content. In contrast, solitary wave packets remain confined to a narrow, nearly vertical region with no appreciable slope.

The spectrograms in Fig. 6(c) and (d) illustrate the effect of nonlinearity at the input wavenumber $\kappa = \frac{\pi}{2}$ and clearly reveal the presence of optical–acoustic mode coupling. In contrast to the linear case, the wavenumber content exhibits strong distortion and shifts outside the input mode branch. As can be seen from Fig. 6(c), for the hardening nonlinearity, the spectrogram spreads markedly within and beyond the initial burst window, indicating a significant shift of the input wavenumber. A similar trend is observed under softening nonlinearity, as shown in Fig. 6(d), although the spread is less pronounced than in the hardening case. Under hardening, the output consists of both dispersed packets and localized solitary-wave components, whereas softening produces only fragmented dispersive packets. In both cases, the input wave packet at $\kappa = \frac{\pi}{2}$ is redistributed across multiple branches, including both high- and low-wavenumber regions.

Finally, the spectrograms for the nonlinear system at the input wavenumber $\kappa = \frac{7\pi}{9}$ are shown in Fig. 6(e) and (f) for hardening and softening nonlinearities, respectively. Under hardening nonlinearity, Fig. 6(e) shows that the wave-packet energy remains strongly localized near the source region. Despite this localization, a portion of the energy shifts to lower wavenumbers, indicating the emergence of a weak acoustic component and confirming optical–acoustic mode coupling in the presence of nonlinearity. The narrow, nearly vertical high-intensity red region highlights the presence of a solitary wave, reflecting substantial energy trapping due to the hardening stiffness. In contrast, the softening case in Fig. 6(f) exhibits a smoother and more diffuse energy distribution. The spectrogram is dispersive, and the wavenumber content spreads both within and outside the initial burst window. In both nonlinear cases, the input optical wave is partially converted to lower-wavenumber components.

4.3. Spectral characterization of the transient wave packets

Having established the wavenumber evolution in Section 4.2, this subsection examines the joint wavenumber–frequency characteristics of the transient wave packets using 2D-FFT contours. The contours represent the spectral amplitude as a function of κ and Ω for both linear and nonlinear cases. Because dispersion and nonlinearity strongly influence the first-optical modes, the analysis focuses on 2D-FFT results obtained from first-optical mode excitation.

The contours in Fig. 7(a) and (b) correspond to input wavenumbers $\kappa = \frac{\pi}{2}$ and $\kappa = \frac{7\pi}{9}$, respectively, and illustrate the acoustic and optical branches of the linear chain, with the dominant spectral energy located on the optical branch in both cases. From Fig. 7(a) and (b), inputting the first optical mode results in partial conversion into the acoustic branch, leading to the propagation of both acoustic and optical components along the chain. The spectral contours remain thin and well defined, indicating that the wave packets remain essentially undistorted and hence, characterizing the linear nature. As shown in Fig. 7(b), excitation at the short wavelength produces a more stretched 2D-FFT signature than the medium-wavelength case in Fig. 7(a), consistent with stronger dispersion. In both excitations, the dominant spectral energy remains concentrated on the optical mode branch.

The 2D-FFT spectral contours for $\kappa = \frac{\pi}{2}$ under hardening and softening nonlinearities are shown in Fig. 7(c) and (d). For the hardening case, Fig. 7(c) shows that the first-optical-mode excitation produces limited spectral spreading within the medium-wavelength region. The wavenumber–frequency content shifts toward the boundaries of the first optical branch, accompanied by energy transfer to the acoustic branch. These effects result in a wider pseudo-bandgap in the first optical mode at medium wavelengths. In contrast, softening nonlinearity leads to dispersive propagation in both acoustic and optical branches, with the dominant spectral amplitude shifted to the acoustic mode, as shown in Fig. 7(d). Previous studies have reported that down-conversion from higher to lower frequencies typically produces a lower-amplitude response, whereas in this softening case the converted acoustic component exhibits higher amplitude, indicated by the red region in Fig. 7(d). The pseudo-bandgap also shifts toward the short-wavelength region. These results demonstrate that both hardening and softening nonlinearities enable tunability of the pseudo-bandgap when the system is excited in the first optical mode.

Finally, the 2D-FFT contours for $\kappa = \frac{7\pi}{9}$ under hardening and softening nonlinearities are shown in Fig. 7(e) and (f). For the hardening case, Fig. 7(e) shows both dispersive and solitary-wave components, where the dispersive component appears as a low-amplitude spread and the solitary wave forms a narrow, high-intensity ridge. It should be noted that solitary-wave signatures are visible along both the acoustic and optical branches, with the dominant spectral amplitude concentrated on the solitary-wave component of the first optical branch. In contrast, the softening case in Fig. 7(f) exhibits only dispersive behavior without solitary-wave formation. The spectral content spreads into the medium-wavelength region of the first optical branch, and the dominant frequency remains near the initial burst under short-wavelength excitation.

The spectrogram and 2D-FFT analyses demonstrate that, unlike earlier systems where nonlinear responses were confined to a single branch [19,29], the proposed metamaterial exhibits simultaneous excitation of the acoustic and optical branches, leading to effective spectral broadening. It should be noted that bandwidth broadening refers to an effective expansion of the accessible propagating spectral range induced by nonlinear optical–acoustic mode coupling. The 2D-FFT results show that a first-optical-mode excitation can generate spectral content on both the optical and acoustic branches, thereby activating additional propagation channels beyond the initially excited branch. Thus, the broadened response arises from inter-branch energy redistribution and multi-branch excitation rather than from a simple increase in the width of an individual dispersion branch. This concurrent activation motivates a detailed investigation of the optical–acoustic coupling mechanism in the next section.

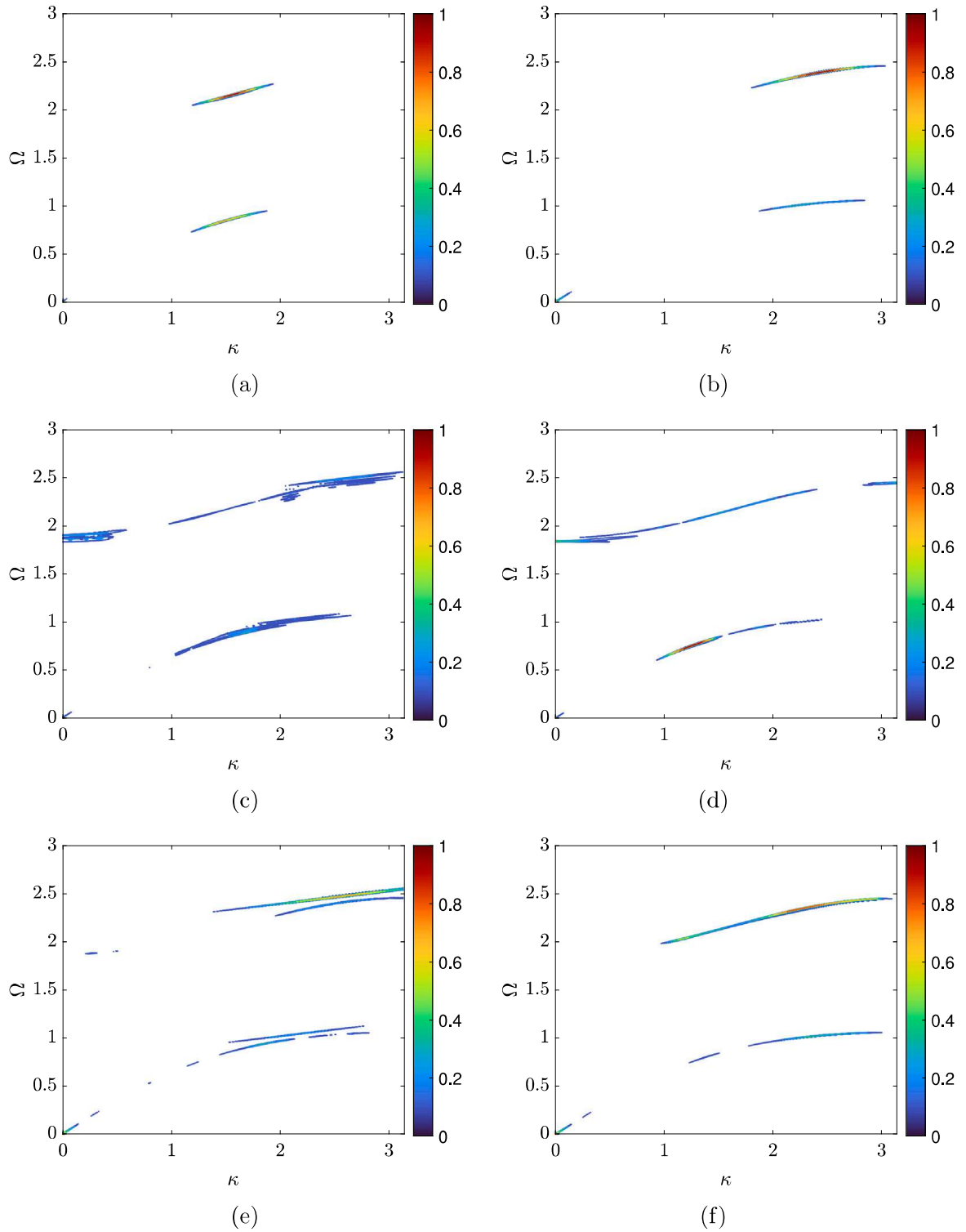


Fig. 7. The 2D-FFT spectral contour illustrates the formation of optical and acoustic modes [49]: (a) linear, $\kappa = \frac{\pi}{2}$, (b) linear, $\kappa = \frac{7\pi}{9}$, (c) hardening, $\kappa = \frac{\pi}{2}$, (d) softening, $\kappa = \frac{\pi}{2}$, (e) hardening, $\kappa = \frac{7\pi}{9}$ and (f) softening, $\kappa = \frac{7\pi}{9}$.

5. Optical-acoustic coupling

In this section, we analyze the optical–acoustic mode coupling observed earlier by examining the relative motions of the outer mass and the local resonators in the hierarchical systems. The chain is excited by wave packets corresponding to the first optical mode at long, medium, and short wavelengths, and these relative motions characterize the propagation of the first optical branch. For the nonlinear case, only the hardening response is presented for brevity, as the softening response exhibits comparable behavior.

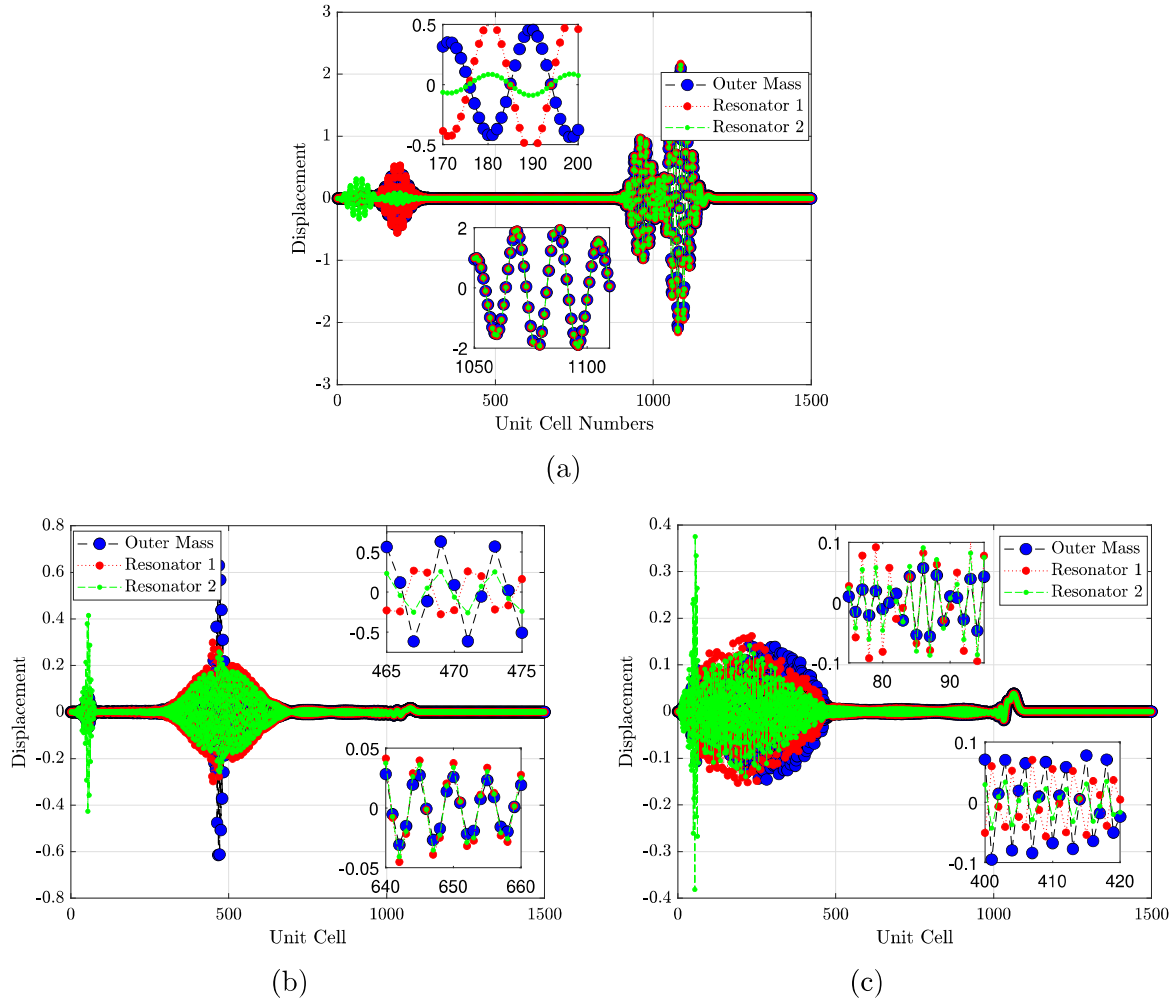


Fig. 8. The relative motion of outer mass and resonators at the end of the simulation excited by first optical mode: (a) linear chain, $\kappa = \frac{\pi}{9}$, (b) linear chain, $\kappa = \frac{\pi}{2}$, and (c) linear chain, $\kappa = \frac{7\pi}{9}$.

Figs. 8 and 9 show the relative displacements for the linear and nonlinear hardening chains, respectively. Specifically, Fig. 8(a)–(c) present the long-, medium-, and short-wavelength excitations for the linear chain, while Fig. 9(a)–(c) show the corresponding responses for the nonlinear hardening chain. In both cases, optical–acoustic mode coupling arises from the spatial variation in the relative displacement as a result of the hierarchical configuration as the wave propagates along the chain.

For the linear chain, Fig. 8(a) shows that the relative displacement of resonator 1 is out of phase with the corresponding outer mass between unit cells 170 and 200; this out-of-phase motion characterizes propagation on the first optical-mode branch. Farther along the chain, for instance between unit cells 1050 and 1100, the relative motion of all masses becomes in phase, indicating the formation of the acoustic-mode branch. It is also noted that, in the linear chain, the outer mass exhibits a larger amplitude when it moves in phase with the local resonators. Similar observations are obtained for the medium- and short-wavelength excitations, as shown in the close-up views of Fig. 8(b) and (c).

For the nonlinear hardening chain, a similar sequence of phase transitions is observed across the long-, medium-, and short-wavelength excitations. The relative motion begins out of phase and gradually transitions to in-phase behavior as the wave propagates along the chain. Under long-wavelength excitation, both optical- and acoustic-mode branches appear, with the optical component exhibiting smaller amplitude and the acoustic component larger amplitude, as shown in Fig. 9(a). For the medium-wavelength case, the relative displacement remains out of phase between unit cells 80 and 140, while an in-phase region forms near unit cells 200–215, as seen in Fig. 9(b). In contrast, for short-wavelength excitation, the outer mass remains out of phase with the resonators over the region of interest, particularly between unit cells 175 and 185, as illustrated in Fig. 9(c). The mode coupling arises from the transition between in-phase and out-of-phase relative motion of the outer mass and the local resonators due to the hierarchical configuration. The hierarchy introduces an additional internal degree of freedom, which enables energy exchange between these modes as the wave propagates along the chain, leading to the observed optical–acoustic mode coupling.

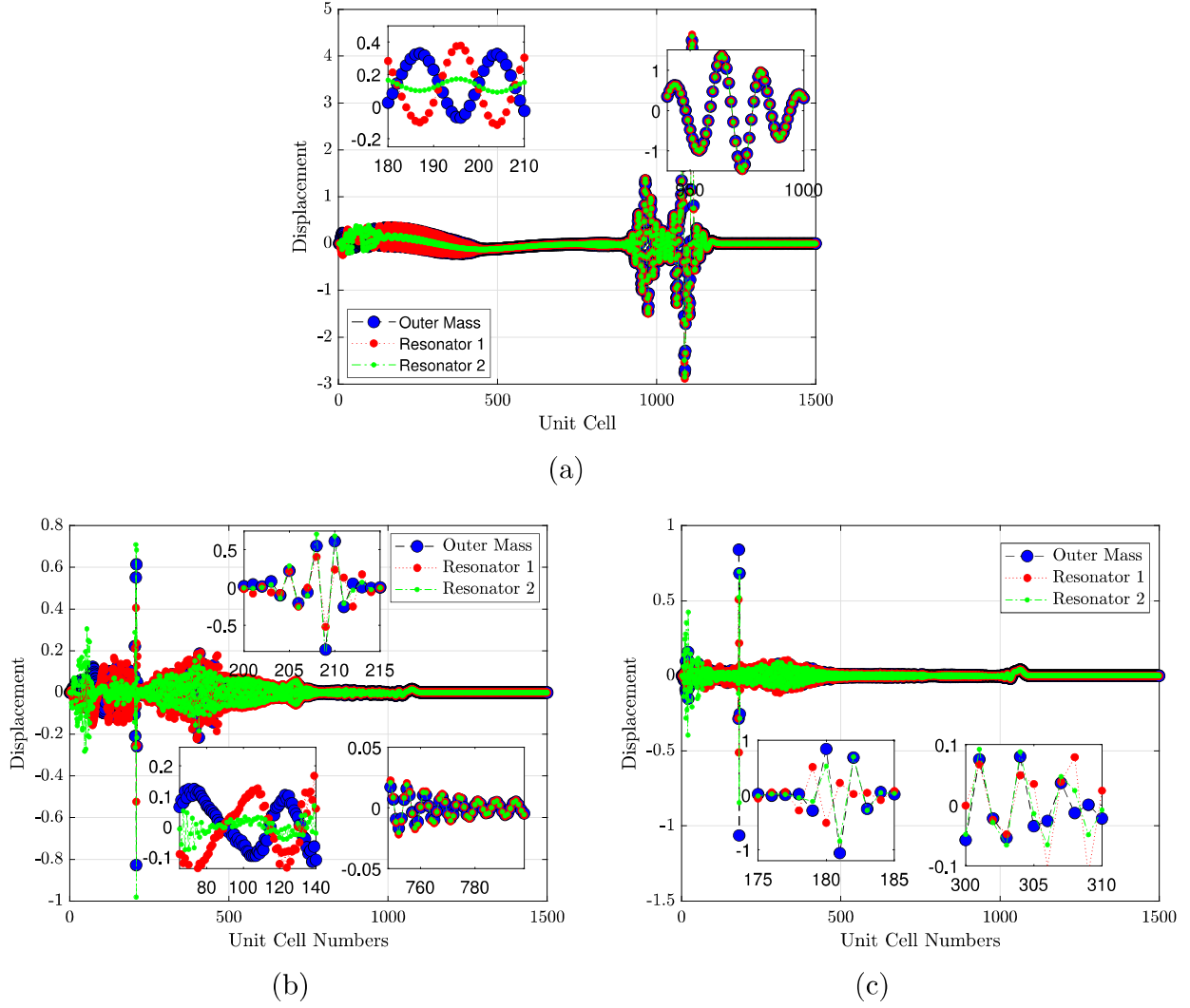


Fig. 9. The relative motion of outer mass and resonators at the end of the simulation excited by first optical mode: (a) nonlinear chain, $\kappa = \frac{\pi}{9}$, (b) nonlinear chain, $\kappa = \frac{\pi}{2}$ and (c) nonlinear chain, $\kappa = \frac{7\pi}{9}$.

To supplement these observations, Fig. 10 presents the FFT of the response under acoustic- and optical-mode excitations. The plots correspond to a 1500-element chain excited at the first unit cell by the normalized force given by

$$f_{ex} = \frac{A_0}{2} \left(H(t) - H\left(t - \frac{2\pi}{\omega} N_{cy}\right) \right) \left(1 - \cos\left(\frac{\omega}{N_{cy}} t\right) \right) \sin(\omega t), \quad (15)$$

with the opposite end left free. The input frequencies are selected within the acoustic and optical branches, corresponding to the normalized values 0.75 and 2.1, respectively. Here, the Hann-modulated spatial window comprises $N_{cy} = 60$ carrier cycles. For the linear chain, the acoustic excitation produces a peak at the input frequency, as shown in Fig. 10(a). Under first-optical-mode excitation, two peaks appear one near the acoustic branch (around $\Omega = 0.5$) and the other in the optical branch as shown in Fig. 10(b), indicating partial conversion of the optical input mode.

For the nonlinear hardening chain, optical-acoustic mode coupling is present under both excitations. In Fig. 10(a), the acoustic input generates components in both the acoustic and optical branches. Likewise, in Fig. 10(b), the first-optical-mode input produces spectral content in both branches, with a portion converted into the acoustic mode. This frequency conversion redistributes energy across a wider spectral range, extending the response toward both higher and lower frequencies and increasing the utilized (or effective) spectrum of the propagating wave. As shown in Fig. 10, the input signal exhibits a relatively narrow and well-defined spectral peak, whereas the nonlinear response shows spectral spreading, the emergence of additional frequency components, and redistribution of energy away from the original excitation frequency. Consequently, the output spans a wider frequency range, not because a single resonance or passband is widened, but because multiple frequencies and branches are activated through nonlinear coupling. This behavior originates from inter-branch energy transfer enabled by the hierarchical coupling between optical and acoustic modes. The observed bandwidth broadening reflects an effective expansion of the usable spectral range, rather than a simple increase in the width of an individual passband.

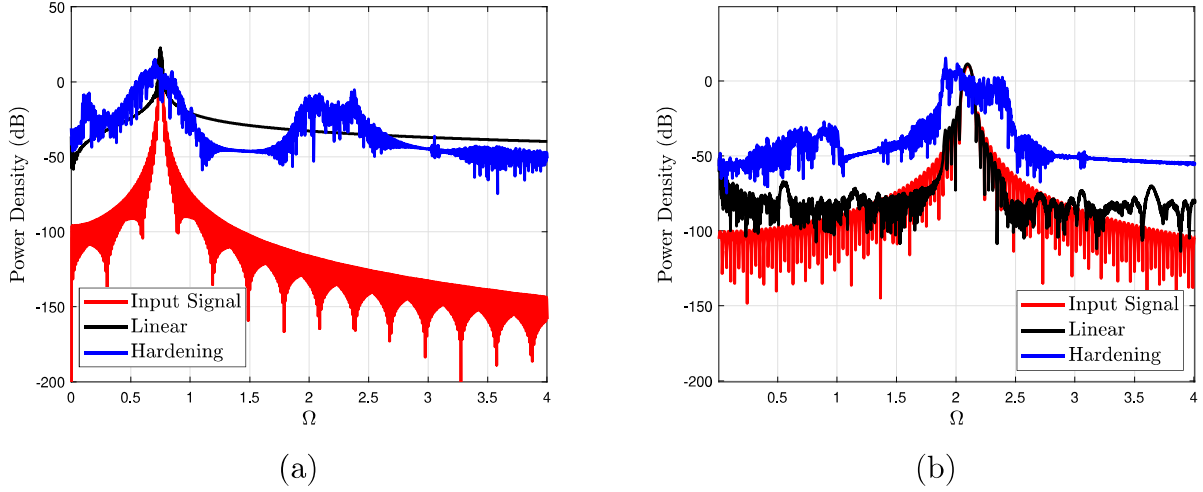


Fig. 10. FFT of the response illustrating an optical-acoustic mode coupling (a) acoustic and (b) optic excitation.

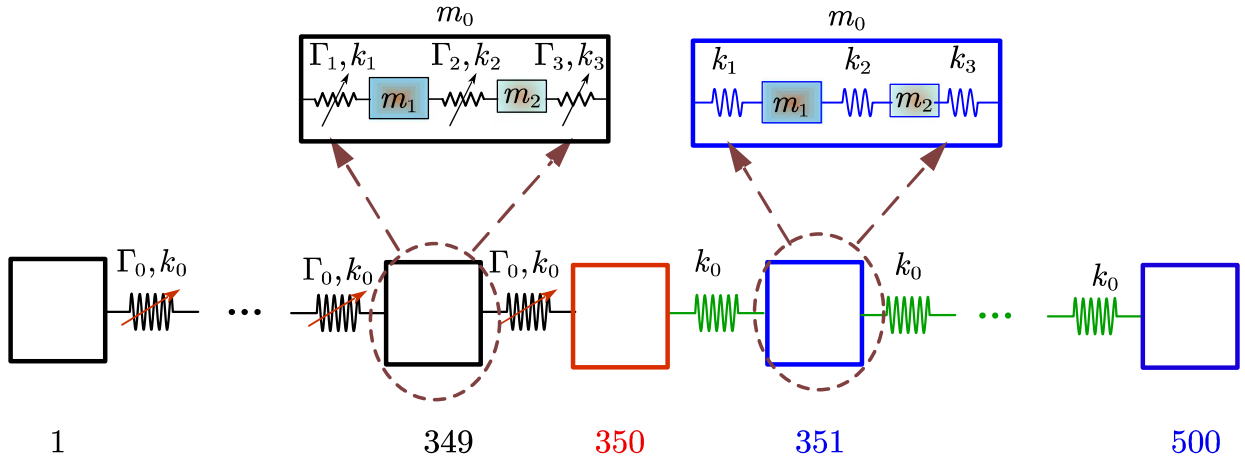


Fig. 11. The schematic diagram of the proposed mechanical diode.

6. Mechanical diode for acoustic wave-guides

Building on the optical–acoustic mode coupling and the amplitude-dependent frequency shifts established in earlier sections, we now propose a mechanical diode based on an asymmetric hybrid linear–nonlinear hierarchy. Structural asymmetry can be systematically introduced through various mechanisms, including nonlinearity variation in the chain [36], geometric disparities [12] and the incorporation of differing mass distributions [44]. The structural asymmetry in the diode, shown in Fig. 11, consists of a nonlinear segment followed by a linear segment such that the pseudo-bandgap of the nonlinear chain overlaps with the bandgap of the linear chain. The configuration comprises 500 unit cells [19], consisting of 349 nonlinear cells, 150 linear cells, and a transition unit cell. Each unit cell incorporates the same local-resonator arrangement described in Fig. 1. The diode is excited using the normalized force f_{ex} defined in Eq. (15), with the excitation frequency selected on the first optical-mode branch within the pseudo-bandgap region indicated in Fig. 13. The excitation is a finite-duration broadband input force (windowed sinusoidal signal), which activates multiple frequency components and enables observation of hierarchical coupling between acoustic and optical branches under transient loading conditions.

The direction-dependent transmission is driven by the combined effect of structural asymmetry and nonlinearity. To demonstrate the directional behavior of the diode, we evaluate its response under forward and reverse excitation. Excitation applied at the nonlinear end corresponds to the forward configuration, as shown in Fig. 12(a), whereas excitation applied at the linear end corresponds to the reverse configuration, as illustrated in Fig. 12(b). Furthermore, to illustrate the design flexibility of the proposed diode, we consider two design configurations obtained by varying the total mass ratio of the local resonators to the mass of the unit cell in the linear segment, denoted by η_l .

In Design A, the mass ratio is set to $\eta_l = 0.55$, which aligns the nonlinear pseudo-bandgap with the broad linear bandgap between the acoustic and first optical branches, as shown in the upper panel of Fig. 13. In the forward excitation, the input signal shifts toward both boundaries of the first optical mode and partially converts into the acoustic branch. In the reverse excitation, the excitation lies entirely within the bandgap, thereby suppressing transmission.

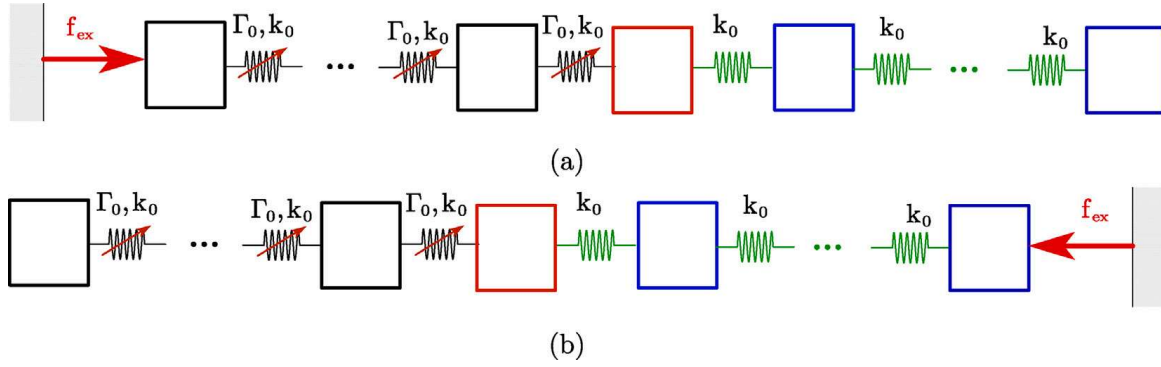


Fig. 12. (a) The forward configuration excitation and (b) the reverse configuration excitation.

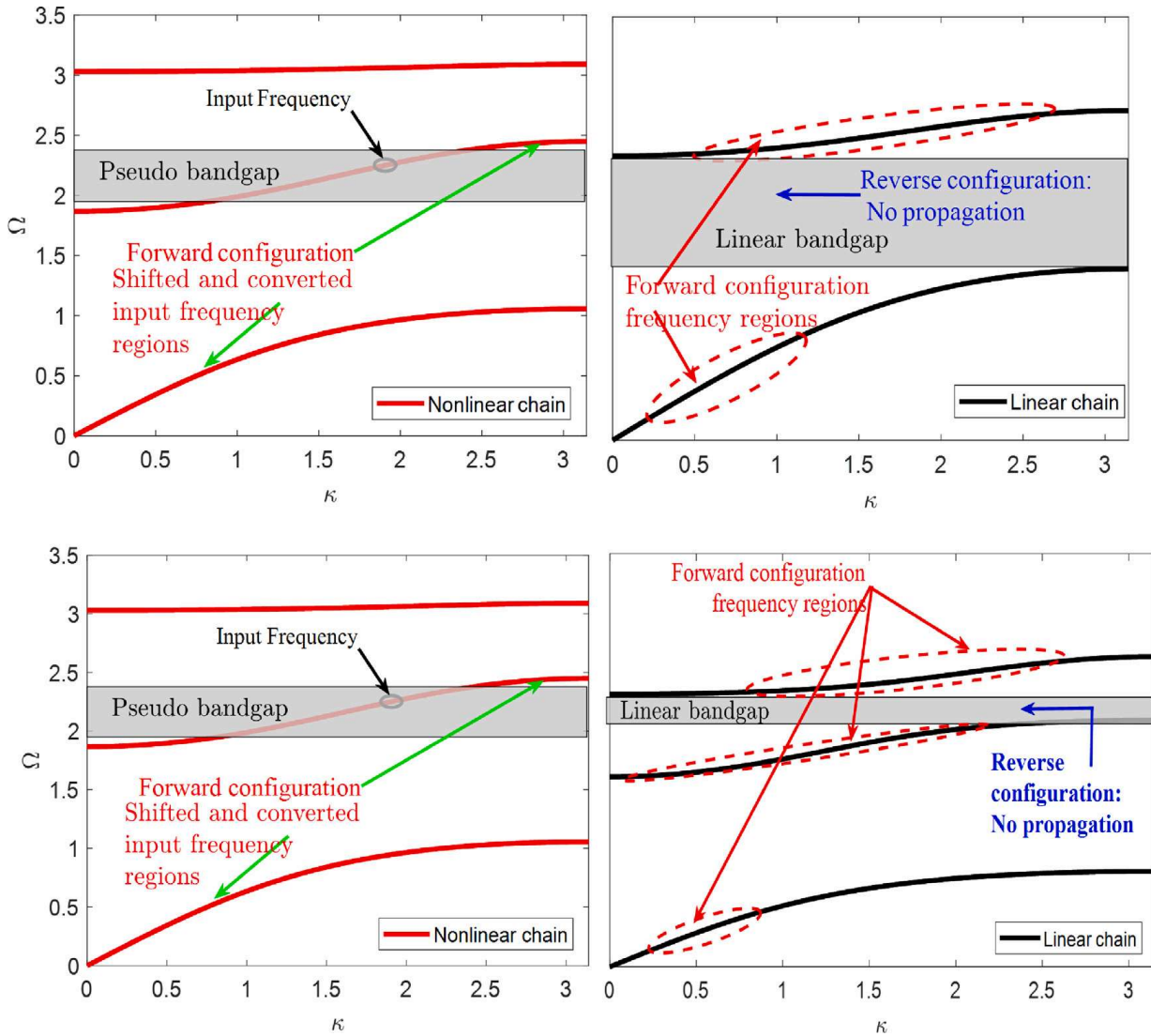


Fig. 13. Conceptual illustration of the first design (top panel at $\eta_l = 0.55$) and second design (bottom panel at $\eta_l = 1.8$) methods for the mechanical diode.

In Design B, we set $\eta_l = 1.8$, yielding overlap between the nonlinear pseudo-bandgap and the narrow linear bandgap between the first and second optical branches, as shown in the lower panel of Fig. 13. All other linear-chain parameters are identical to those of the nonlinear chain for both designs. It should be noted that Design B provides a wider forward-propagation bandwidth than Design A due to amplitude-dependent frequency shifts toward the lower boundary of the first optical mode and toward the second optical mode. This behavior is advantageous for waveguiding applications requiring selective multi-channel transmission.

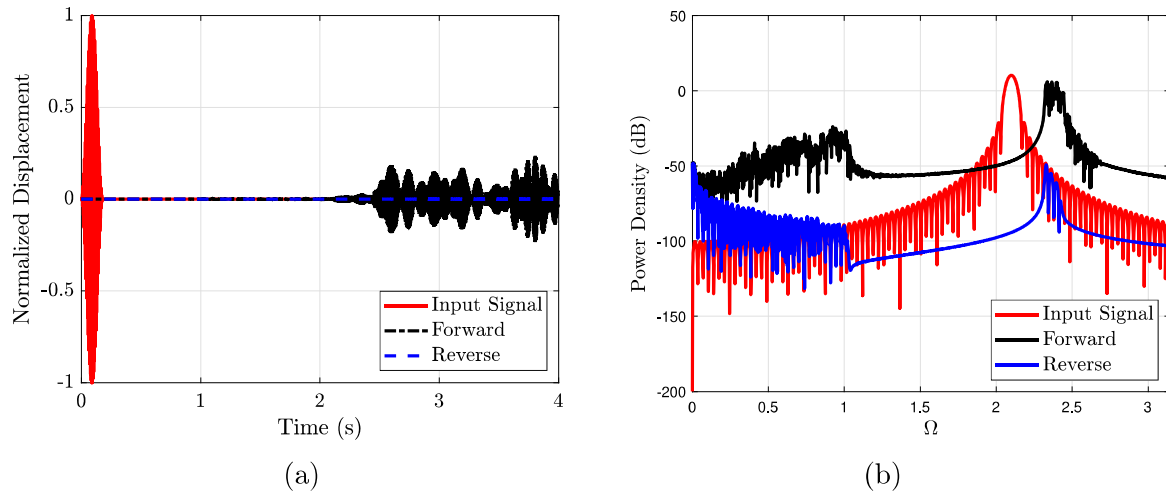


Fig. 14. The forward and reverse response of a Design A, hardening nonlinear chain and linear chain with $\eta_l = 0.55$, (a) Time response and (b) FFT of input signal and responses.

It should be noted here, that in contrast to the approach of Zhou et al. [19], which required simultaneous tuning of multiple parameters in the local resonators, the present designs achieve directional bias by adjusting only the total mass ratio of the resonators in the linear segment. This highlights the simplicity and robustness of the hierarchical architecture for mechanical-diode design. In the analysis we explore both hardening- and softening-type nonlinearities in the nonlinear segment for the forward and reverse configurations described earlier.

The effectiveness of wave transmission is quantified using the transmission ratio σ and the asymmetry ratio AR . The transmission ratio σ is defined as the ratio of the total output energy at the last unit cell to the total input energy at the first unit cell, obtained by integrating the normalized power at the input and output over the signal duration [28]:

$$\sigma = \frac{\int_0^{\tau_f} P_{end}(\tau) d\tau}{\int_0^{\tau_f} P_{in}(\tau) d\tau}. \quad (16)$$

Here, P_j denotes the normalized power of the j th outer mass, given by [19]

$$P_j = -\left[(u_j - u_{j-1}) + \epsilon\gamma_0(u_j - u_{j-1})^3\right] \frac{\partial u_j}{\partial \tau}. \quad (17)$$

Note that for the linear segment of the diode, the cubic nonlinear term in Eq. (17) is omitted. The degree of directional asymmetry is quantified using the asymmetry ratio

$$AR = \frac{\sigma_{forward}}{\sigma_{reverse}},$$

which provides a quantitative measure of directional wave transmission.

Figs. 14 and 15 present the time- and frequency-domain responses of the diode for Design A and Design B, respectively, under hardening nonlinearity. In both designs, forward excitation produces measurable transmission, whereas the reverse response remains minimal, demonstrating strong directional selectivity. The forward-transmitted signal is stretched while the reverse signal is attenuated, as shown in Figs. 14(a) and 15(a). For Design A, the first-optical-mode input partially converts into the acoustic branch and shifts toward the upper boundary of the passband, as illustrated in Fig. 14(b). This coupled acoustic–optical propagation broadens the spectral content and distributes energy across both lower- and higher-frequency regions. In Design B, optical–acoustic mode coupling is facilitated by a shift of the first-optical-mode frequency toward its lower boundary, consistent with the nonlinear tuning of the pseudo-bandgap, as shown in Fig. 15(b).

For Design A, the forward configuration yields a transmission ratio of $\sigma = 0.2709$, whereas the reverse configuration leads to $\sigma = 5.0394 \times 10^{-7}$. This corresponds to an asymmetry ratio of $AR = 5.3761 \times 10^5$, further indicating an efficient forward propagation and near-complete suppression of reverse transmission. For Design B, the forward transmission ratio is $\sigma = 0.4203$, reflecting the broader forward-propagation spectrum obtained by shifting the first-optical-mode response toward lower frequencies. The reverse transmission ratio is $\sigma = 2.4876 \times 10^{-5}$, resulting in $AR = 1.6894 \times 10^4$.

Finally, Figs. 16 and 17 show the forward and reverse responses of the proposed mechanical diode under softening nonlinearity for Design A and Design B, respectively. We observe that the overall time- and frequency-domain characteristics remain consistent with those observed in the hardening case. For Design A, the forward and reverse transmission ratios are $\sigma = 0.3682$ and $\sigma = 6.4418 \times 10^{-6}$, respectively, yielding an asymmetry ratio of $AR = 5.7163 \times 10^4$. This further imply that the forward transmission ratio for Design A under softening nonlinearity is approximately 35.9% higher than that obtained with hardening nonlinearity under the same system parameters. For Design B, the softening nonlinear chain produces a forward transmission ratio of $\sigma = 0.5367$ and a

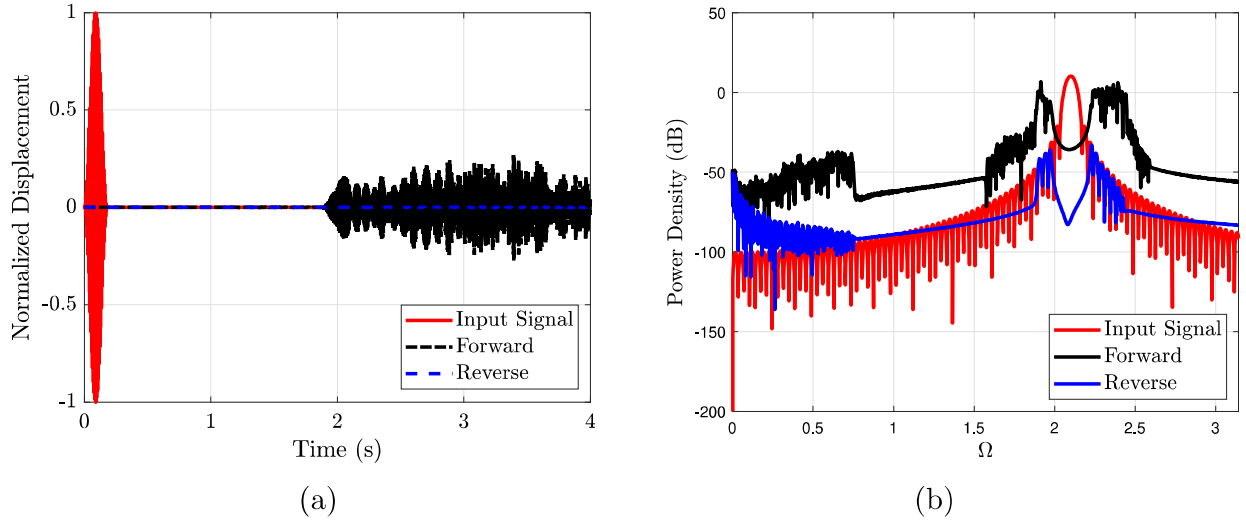


Fig. 15. The forward and reverse response of a Design B, hardening nonlinear chain and linear chain with $\eta_l = 1.8$, (a) time response and (b) FFT of input signal and responses.

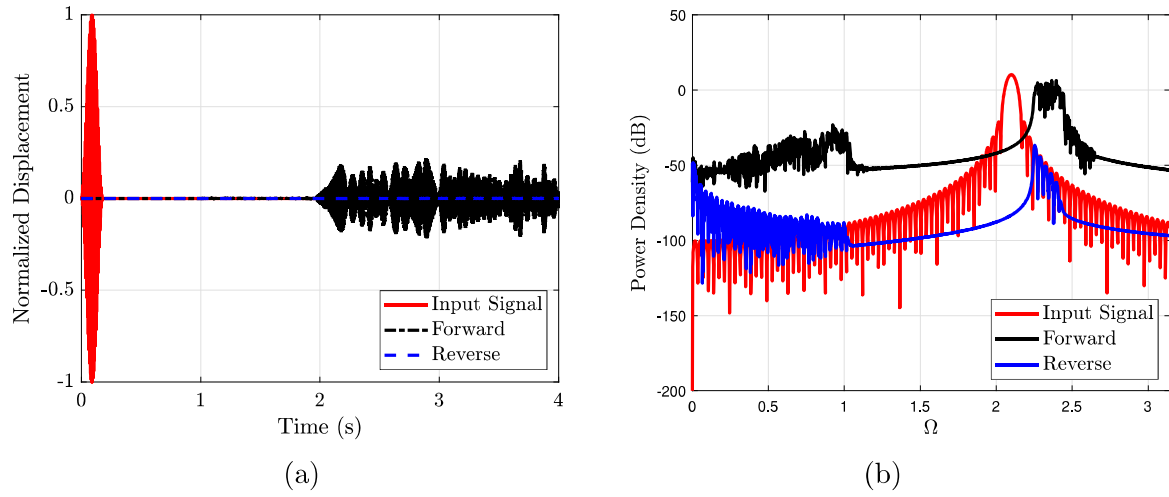


Fig. 16. The forward and reverse response of Design A, softening nonlinear chain and linear chain with $\eta_l = 0.55$, (a) time response and (b) FFT of input signal and responses.

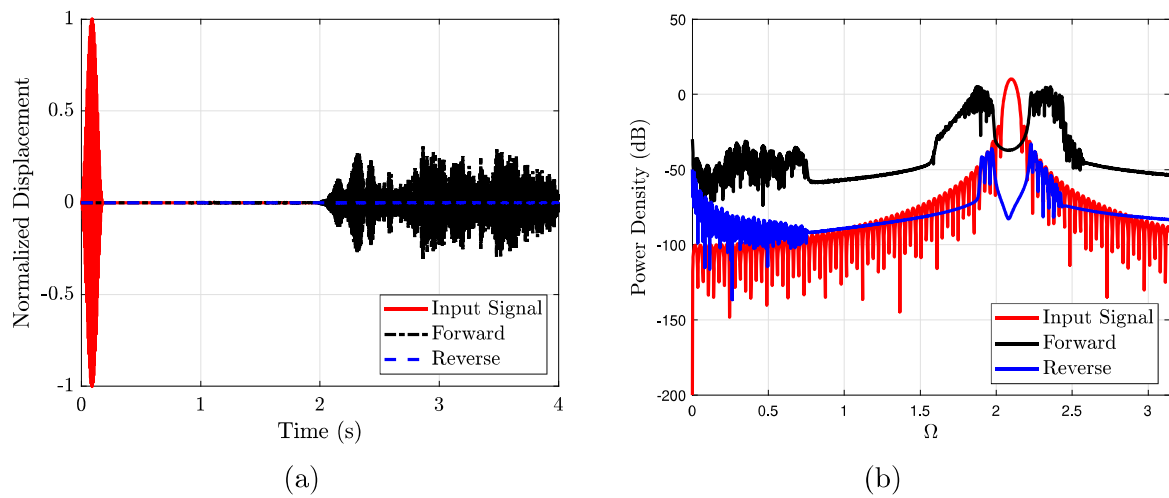


Fig. 17. The forward and reverse response of Design B, softening nonlinear chain and linear chain with $\eta_l = 1.8$, (a) time response and (b) FFT of input signal and responses.

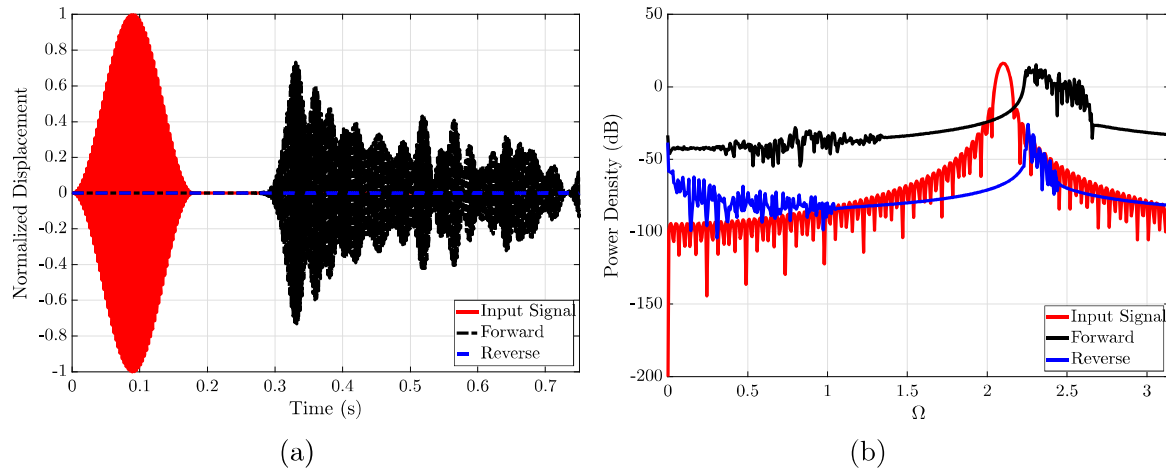


Fig. 18. The forward and reverse response of a Design A, hardening nonlinear chain and linear chain with 70 unit cells and $\eta_l = 0.55$, (a) Time response and (b) FFT of input signal and responses.

reverse transmission ratio of $\sigma = 2.4875 \times 10^{-5}$, corresponding to $AR = 2.1575 \times 10^4$. Based on forward transmission alone, Design B with softening nonlinearity is the most efficient, although its asymmetry ratio is comparatively lower.

To this end, both hardening and softening nonlinear chains offer distinct advantages within the two diode configurations. Across both designs, Design B yields higher forward transmission ratios, with the softening chain providing the greatest transmission efficiency. In contrast, the hardening chain achieves larger asymmetry ratios, indicating stronger directional asymmetry. These outcomes reflect a trade-off between maximizing forward transmission and enhancing directional asymmetry. Compared to conventional locally resonant metamaterials (i.e., single-resonator systems), the proposed hierarchical configuration achieves substantially lower reverse transmission, reaching values as low as 5.04×10^{-7} , and a higher asymmetry ratio of up to 5.3761×10^5 . In addition, it exhibits an improved balance between operational frequency range and structural scale, thereby alleviating the low-frequency–thickness trade-off. To quantitatively assess this trade-off, we define an effective transmission bandwidth as the total frequency range of the acoustic and optical mode branches for which the transmission exceeds -50 dB. Based on this definition, the bandwidths of the proposed diodes are 0.89 and 1.25 for hardening nonlinearity in Designs A and B, respectively, and 1.07 and 1.26 for softening nonlinearity in Designs A and B, respectively. Design B exhibits a larger spectral width due to redistribution of input energy into the acoustic and first two optical branches of the linear chain, consistent with the conceptual design shown in Fig. 13(b). Under the same conditions, the conventional nonlinear locally resonant metamaterial gives a narrower bandwidth of 0.81 due to isolated and limited low-frequency passbands [19].

The nonlinear hierarchical mechanical diode is initially demonstrated using a chain consisting of 500 unit cells. This relatively large system size is intentionally adopted to capture nonlinear wave propagation, minimize boundary effects, and enable the full development of acoustic and optical mode branches. To examine the potential for compact implementation, we also analyzed a reduced system with 70 unit cells while maintaining the same proportion of nonlinear elements. The response of the system with a hardening nonlinear chain of Design A is illustrated in Fig. 18. The results show that the diode functionality is preserved, with transmission coefficients of 0.8443 (forward) and 1.4614×10^{-5} (reverse), corresponding to an asymmetry ratio of 5.78×10^4 . Furthermore, the spectral analysis reveals the presence of both acoustic and optical mode branches in the reduced system (Fig. 18(b)), indicating that the underlying nonlinear mode-coupling mechanism is preserved in the smaller system. These results suggest that the proposed design principle can be extended toward more compact implementations.

This hierarchical metamaterial is also suitable for other applications such as vibration isolation, and energy harvesting. Under the same forcing condition as the diodes, where energy is distributed across multiple mode branches, the system can be utilized for broadband simultaneous energy harvesting from both acoustic and optical responses. These features also contribute to enhanced vibration attenuation over a wider frequency range, making the proposed system suitable for vibration isolation.

7. Conclusion

In this study, we established a unified analytical–numerical framework to characterize amplitude-dependent dispersion and optical–acoustic interactions in a hierarchical nonlinear acoustic metamaterial. The metamaterial was modeled as a mass–spring chain with inwardly configured local resonators embedded within each host mass. The weakly nonlinear dispersion relation, validated through transient simulations and 2D-FFT analysis, showed that chain and resonator nonlinearities tune distinct branches of the spectrum and jointly enable broadband, amplitude-dependent shifts in bandgaps and mode positions. Wave-packet and spectro-spatial analyses were employed to investigate the resulting propagation behaviors. We observed optical–acoustic mode coupling, nonlinear frequency shifts and down-conversion. The inward hierarchical architecture promoted strong optical–acoustic interactions, with the local motion transitioning between out-of-phase and in-phase states along the chain. This mechanism drove energy redistribution across branches, leading to spectral broadening and enhanced effective propagation bandwidth, thereby

enabling controllable conversion between optical and acoustic modes. Leveraging these mechanisms, two passive mechanical-diode designs were developed by aligning nonlinear pseudo-bandgaps with linear bandgaps. Design A achieved stronger directional asymmetry, whereas Design B provided higher forward transmission efficiency, revealing a practical trade-off between directional isolation and transmissibility. Overall, the analytical and numerical results indicate that hierarchical nonlinear metamaterials can serve as a compact and tunable model platform for studying amplitude-dependent dispersion engineering, broadband mode conversion, and passive direction-dependent transmission. These findings may inform future designs of vibration-isolation, wave-guiding, and energy-flow-management devices, although experimental validation and implementation-specific constraints remain important directions for future work. Future study will focus on extending this work to include experimental validation and assessment under practical implementation constraints.

CRedit authorship contribution statement

Ashenafi Abebe Mebrat: Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Joshua LeGrande:** Writing – review & editing, Validation. **Sunit K. Gupta:** Writing – review & editing, Validation, Supervision, Investigation. **Oumar Barry:** Writing – review & editing, Supervision, Resources, Project administration, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Expression of constants

$$\begin{aligned}
 C_0 &= A_0 \frac{\alpha_2 K_{\omega 0} + \alpha_3}{\alpha_2 + \alpha_3 - \eta \omega_0^2 (1 - \beta_1)}, B_0 = A_0 \frac{(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 - \eta \alpha_1 \omega_0^2 (1 - \beta_1))}{K_{\omega 0}} \\
 K_{\omega 0} &= \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 - \eta \omega_0^2 (\alpha_1 + \alpha_2 - \beta_1 (\alpha_1 + \alpha_3)) + \eta^2 \beta_1 \omega_0^4 (1 - \beta_1) \\
 R_1 &= -2\Omega_0 \Omega_1 \frac{\partial^2 u_j^0}{\partial \tau^2} - \gamma_0 [(u_j^0 - u_{j-1}^0)^3 + (u_j^0 - u_{j+1}^0)^3] - \gamma_1 (u_j^0 - y_{j1}^0)^3 - \gamma_3 (u_j^0 - y_{j2}^0)^3 \\
 R_2 &= -2\eta \beta_1 \Omega_0 \Omega_1 \frac{\partial^2 y_{j1}^0}{\partial \tau^2} - \gamma_1 (y_{j1}^0 - u_j^0)^3 - \gamma_2 (y_{j1}^0 - y_{j2}^0)^3 \\
 R_3 &= -2\eta (1 - \beta_1) \Omega_0 \Omega_1 \frac{\partial^2 y_{j2}^0}{\partial \tau^2} - \gamma_2 (y_{j2}^0 - y_{j1}^0)^3 - \gamma_3 (y_{j2}^0 - u_j^0)^3 \\
 P &= \eta (1 - \beta_1) \Omega_0^2 \frac{\partial^2}{\partial \tau^2} + \alpha_2 + \alpha_3, Q = P(\eta \beta_1 \Omega_0^2 \frac{\partial^2}{\partial \tau^2} + \alpha_1 + \alpha_2) - \alpha_2^2, \Omega_1 = \frac{X}{Y} \\
 X &= 3(\alpha_1 + \alpha_3 + \eta(-1 + \beta_1)\omega_0^2) \left[\alpha_1^2 \left(16A_0^3 \sin^4(k/2)\gamma_0 + (B_0 - C_0)^3\gamma_2 - (A_0 - C_0)^3\gamma_3 \right) \right. \\
 &\quad + \alpha_2^2 \left(-16A_0^3 \sin^4(k/2)\gamma_0 - (A_0 - B_0)^3\gamma_1 + (A_0 - C_0)^3\gamma_3 \right) \\
 &\quad + \alpha_2 \left(2\alpha_3 \left(8A_0^3 \sin^4(k/2)\gamma_0 - (A_0 - C_0)^3\gamma_3 \right) \right. \\
 &\quad \left. + \eta(-1 + \beta_1) \left(16A_0^3 \sin^4(k/2)\gamma_0 + (A_0 - B_0)^3\gamma_1 - (A_0 - C_0)^3\gamma_3 \right) \omega_0^2 \right) \\
 &\quad + \eta \beta_1 \omega_0^2 \left[\alpha_3 \left(-16A_0^3 \sin^4(k/2)\gamma_0 - (A_0 - B_0)^3\gamma_1 + B_0^3\gamma_2 \right. \right. \\
 &\quad \left. \left. - 3B_0^2 C_0 \gamma_2 + 3B_0 C_0^2 \gamma_2 - C_0^3 \gamma_2 + 2A_0^3 \gamma_3 - 6A_0^2 C_0 \gamma_3 + 6A_0 C_0^2 \gamma_3 - 2C_0^3 \gamma_3 \right) \right. \\
 &\quad \left. + \eta(-1 + \beta_1) \left(-16A_0^3 \sin^4(k/2)\gamma_0 - (A_0 - B_0)^3\gamma_1 + (A_0 - C_0)^3\gamma_3 \right) \omega_0^2 \right] \\
 &\quad + \alpha_1 \left[2\alpha_3 \left(8A_0^3 \sin^4(k/2)\gamma_0 - (A_0 - C_0)^3\gamma_3 \right) + \alpha_2 \left(16A_0^3 \sin^4(k/2)\gamma_0 + (A_0 - B_0)^3\gamma_1 \right. \right. \\
 &\quad \left. \left. - B_0^3\gamma_2 + 3B_0^2 C_0 \gamma_2 - 3B_0 C_0^2 \gamma_2 + C_0^3 \gamma_2 - 2A_0^3 \gamma_3 + 6A_0^2 C_0 \gamma_3 - 6A_0 C_0^2 \gamma_3 + 2C_0^3 \gamma_3 \right) \right. \\
 &\quad \left. + \eta \left(-16A_0^3 \sin^4(k/2)\gamma_0 - B_0^3\gamma_2 + 3B_0^2 C_0 \gamma_2 - 3B_0 C_0^2 \gamma_2 + C_0^3 \gamma_2 \right. \right. \\
 &\quad \left. \left. - \beta_1 \left((A_0 - B_0)^3\gamma_1 - (B_0 - C_0)^3\gamma_2 \right) + A_0^3\gamma_3 - 3A_0^2 C_0 \gamma_3 + 3A_0 C_0^2 \gamma_3 - C_0^3 \gamma_3 \right) \omega_0^2 \right] \Big]
 \end{aligned}$$

$$\begin{aligned}
Y = & 2\omega_0(\alpha_1 + \alpha_3 + \eta(-1 + \beta_1)\omega_0^2) \left[-A_0\alpha_2^2 + \alpha_1^2(A_0 + B_0\eta\beta_1) + \alpha_2(\alpha_3(A_0 + C_0\eta \right. \\
& + (B_0 - C_0)\eta\beta_1) + A_0\eta(-1 + \beta_1)\omega_0^2) - \eta\beta_1\omega_0^2(\alpha_3(A_0 + C_0\eta - C_0\eta\beta_1) \\
& + A_0\eta(-1 + \beta_1)\omega_0^2) + \alpha_1(\alpha_3(A_0 + C_0\eta + (B_0 - C_0)\eta\beta_1) + \alpha_2(A_0 + C_0\eta - C_0\eta\beta_1) \\
& \left. + \eta(-A_0 - B_0\eta\beta_1 + B_0\eta\beta_1^2)\omega_0^2) \right]
\end{aligned}$$

Data availability

Data will be made available on request.

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