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FREE VIBRATION ANALYSIS OF A BEAM UNDER AXIAL LOAD CARRYING A MASS-SPRING-MASS

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ABSTRACT

This paper deals with the free vibration analysis of a beam subjected to an axial tensile load with an attached in-span mass-spring-mass system. The equations of motion are derived by means of the Hamilton principle and an explicit expression of the frequency equation is presented. The formulation is validated with results in the literature and the finite element method. Parametric studies are done to investigate the effect of the axial load, the magnitude and location of the mass-spring-mass system on the lowest five natural frequencies and mode shapes. The results indicate that the fundamental mode is independent of the tension and the in-span mass. However, a significant change in all modes is observed when the position of the mass- spring-mass is varied.

NOMENCLATURE

M_c In-span mass
M_d Free Mass
k Spring stiffness
$W(x,t)$ Transverse displacement of the beam
$z(t)$ Transverse displacement of the free mass
ω Natural frequency of the loaded beam
ξ Non dimentional coordinate in the longitudinal direction of the beam

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INTRODUCTION

The transverse vibration of a beam under tensile axial loading carrying an mass-spring-mass system is of practical interest. It has a wide range of application in civil, mechanical, and aerospace engineering. In the design of high-voltage transmission lines, for example, the cable is modeled as a tensile Euler-Bernoulli beam and the Stockbridge damper can be represented as a mass-spring-mass system.

Numerous researchers have examined the free vibration analysis of a beam carrying concentrated masses. Hamdan et al. [1] employed the Dirac delta function to analytically solve the free vibration and forced vibration of a cantilever beam carrying a concentrated mass. Chen [2] also used a Dirac delta function to study the vibration of a simply supported beam that is carrying a concentrated mass. Naguleswaran [3] obtained the exact natural frequency of an Euler-Bernoulli beam carrying two in-span particles. Ercoli et al. [4] numerically and experimentally investigated the fundamental mode of a beam carrying elastically mounted masses. Using Laplace transform, Liu et al. [5] examined the free vibration of a beam hinged by a rotational spring at one end and carrying a concentrated mass at the other end. Wu et al. [6] studied the free vibration of a beam carrying numerous rigidly attached point masses using a combined numerical and analytical approach.

Extensive research on the free vibration of a beam carrying spring-mass system has also been reported. Gurgoze [7] inves-

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tigated the frequencies of a cantilevered Euler-Bernouilli beam with spring-mass system attached in-span by means of the Lagrange multiplier method. Cha [8] studied the natural frequencies of a linear structure carrying spring-mass systems using the assumed modes method. Hsien et al. [9] examined the exact natural frequencies of a uniform multi-span beam carrying multiple spring-mass systems.

The majority of the studies reported in the literature focused on the study of a beam with either an in-span mass or a spring-mass system, a series of each type in some cases. However, there is no work that simultaneously examined the free vibration of a beam with an attached in-span mass-spring-mass system subjected to pretension. The investigation of this problem is reported in the present paper. An explicit expression is presented for the frequency equation. Parametric studies are used to examine the effect of the magnitude and location of the in-span mass-spring-mass system on the natural frequency as well as the role of the axial tensile load. The results are validated using the finite element method and results from the literature.

GOVERNING EQUATIONS

The system of interest is schematically depicted in Fig. 1. Two frames of reference (0_1 and 0_2) are employed. An in-span mass, M_c , is attached on the beam at a distance L_d from frame 0_1 . A spring with a stiffness k connects the free mass, M_d, to the in-span mass . The system kinetic and potential energy are given as:

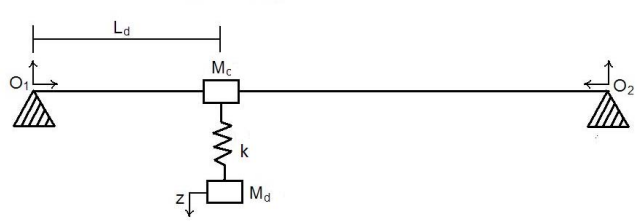

FIGURE 1. SCHEMATIC OF A SIMPLY SUPPORTED BEAM WITH AN IN-SPAN MASS AND SPRING MASS SYSTEM .

$$T = \frac{1}{2}m\int_0^{L_\mathrm{i}} \dot{W}_\mathrm{i}^2(x,t)dx + \frac{1}{2}M_d\dot{z}^2 + \frac{1}{2}M_\mathrm{c}\dot{z_0}^2 \tag{1}$$

$$V = \frac{1}{2}EI\int_0^{L_\mathrm{i}} W_\mathrm{i}''^2(x,t)dx + \frac{1}{2}k(z-z_0)^2 + \frac{1}{2}T\int_0^{L_\mathrm{i}} W_\mathrm{i}'^2(x,t) \tag{2}$$

where $z_0 = W_1(L_\mathrm{d},t)$ denotes the transverse displacement of the in-span mass, M_c. EI denotes the flexural rigidity of the beam, m is the mass per unit length of the beam, and T denotes the tension in the beam. W and z denote the transverse displacement of the beam and the free mass, respectively. The primes and dots represent the derivative with respect to the axial coordinate x and time t, respectively. Using Hamilton principle along with the system energy expressions yields the following system equations

$$EIW_\mathrm{i}^{IV} + m\ddot{W}_\mathrm{i} - TW_\mathrm{i}'' = 0 \quad (\text{for j} = 1,2) \tag{3}$$

$$M_\mathrm{d}\ddot{z} + k(z-z_0) = 0 \tag{4}$$

Assuming that the whole system exhibits harmonic vibration, $W(x,t)$ and $z(t)$ can be expressed as follows:

$$W(x,t) = y(x)e^{i\omega t} \tag{5}$$

$$z(t) = Ae^{i\omega t} \tag{6}$$

where $y(x)$ and A are the amplitudes of $W(x,t)$, and $z(t)$. And ω is the natural frequency of the system. Substituting the above equations into the governing equations of motion yields the non-dimensional system governing equations

$$Y_\mathrm{j}^{IV}(\xi) - s^2Y_\mathrm{j}^{II}(\xi) - \Omega^4Y_\mathrm{j}(\xi) = 0 \tag{7}$$

$$Z - KY_1(\xi_1) = 0 \tag{8}$$

where $Y = \dfrac{y}{L}$, $K = \dfrac{k}{k - M_\mathrm{d}\omega^2}$, $\xi_1 = \dfrac{L_\mathrm{d}}{L}$, $x = \dfrac{\xi}{L}$, $s^2 = \dfrac{TL^2}{EI}$, $Z = \dfrac{A}{L}$, $\Omega^4 = \dfrac{m\omega^2}{EI}L^4$.

FREQUENCY EQUATION AND MODE SHAPES

The shape function of each segment $(\mathrm{j} = 1,2)$ is expressed as

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$$Y_{\mathrm{j}} = c_{\mathrm{j}1}\sin\alpha\xi + c_{\mathrm{j}2}\cos\alpha\xi + c_{\mathrm{j}3}\sinh\beta\xi + c_{\mathrm{j}4}\cosh\beta\xi \qquad (9)$$

where $\alpha = \sqrt{-\frac{s^2}{2} + \sqrt{\frac{s^4}{4} + \Omega^4}}, \quad \beta = \sqrt{\frac{s^2}{2} + \sqrt{\frac{s^4}{4} + \Omega^4}}$

Observing the boundary conditions of the beam yields

$$Y_1(0) = Y_2(0) = Y_1''(0) = Y_2''(0) = 0$$
$$c_{12} = c_{14} = c_{32} = c_{34} = 0$$

Hence Eq. 9 can be written as

$$Y_{\mathrm{j}} = c_{\mathrm{j}1} sin\alpha\xi_{\mathrm{j}} + c_{\mathrm{j}3} sinh\beta\xi_{\mathrm{j}} \quad (\text{for j} = 1,2) \qquad (10)$$

For brevity (i = 1, 2) let

$$\mathrm{s_i} = \sin\alpha\xi_{\mathrm{i}}$$
$$\mathrm{c_i} = \cos\alpha\xi_{\mathrm{i}}$$
$$\mathrm{sh_i} = \sinh\beta\xi_{\mathrm{i}}$$
$$\mathrm{ch_i} = \cosh\beta\xi_{\mathrm{i}}$$

Applying continuity conditions at $\xi = \xi_1$,

$$Y_1(\xi_1) = Y_2(\xi_2)$$
$$Y_1'(\xi_1) = -Y_2'(\xi_2)$$
$$Y_1''(\xi_1) = Y_2''(\xi_2)$$
$$-Y_1'''(\xi_1) + s^2 Y_1'(\xi_1) + \eta Y_1(\xi_1) - \gamma K Y_1(\xi_1) = -Y_2'''(\xi_2) - s^2 Y_2'(\xi_2)$$

where $\eta = \frac{k - \omega^2 M_{\mathrm{c}}}{EI} L^3, \quad \gamma = \frac{kL^3}{EI}$

Substituting Eq. 10 into the above equations, the resulting expressions can be written in matrix format as

$$\begin{bmatrix} s_1 & sh_1 & -s_2 & -sh_2 \\ \alpha c_1 & \beta ch_1 & \alpha c_2 & \beta ch_2 \\ -\alpha^2 s_1 & \beta^2 sh_1 & \alpha^2 s_2 & -\beta^2 sh_2 \\ \phi & \varepsilon & \kappa & \chi \end{bmatrix} \begin{Bmatrix} c_{11} \\ c_{13} \\ c_{21} \\ c_{23} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (11)$$

where

$$\phi = \alpha c1(\alpha^2 + s^2) + s_1(\eta - \gamma K)$$
$$\varepsilon = ch_1\beta(-\beta^2 + s^2) + sh_1(\eta - \gamma K)$$
$$\kappa = c_2\alpha(\alpha^2 + s^2)$$
$$\chi = \beta ch_2(-\beta^2 + s^2)$$

The characteristic (or frequency) equation is now obtained by equating the determinant of the 4x4 matrix in Eq. 11 to zero. It may be written as

$$\begin{aligned}
&(k - \omega^2 M_{\mathrm{d}})(-\beta\eta\alpha^2 c_\alpha \theta sh_\beta \psi + \alpha\eta\beta^2 s_\alpha \psi ch_\beta \psi \\
&- \alpha\eta\beta^2 s_\alpha \psi ch_\beta \theta - \alpha\beta^2\gamma K s_\alpha \psi ch_\beta \psi + \alpha\beta^2\gamma K s_\alpha \psi ch_\beta \theta \\
&+ \beta\eta\alpha^2 c_\alpha \psi sh_\beta \psi - \beta\gamma K\alpha^2 c_\alpha \psi sh_\beta \psi - \alpha^3\gamma K s_\alpha \psi ch_\beta \psi \\
&+ \alpha^3\gamma K s_\alpha \psi ch_\beta \theta - \beta^3\gamma K c_\alpha \psi sh_\beta \psi + \beta^3\eta c_\alpha \psi sh_\beta \psi \qquad (12) \\
&- 2\alpha\beta^5 s_\alpha \psi sh_\beta \psi - 2\alpha^5\beta s_\alpha \psi sh_\beta \psi - \beta^3\eta c_\alpha \theta sh_\beta \psi \\
&- 4\alpha^3\beta^3 s_\alpha \psi sh_\beta \psi + \alpha^3\eta s_\alpha \psi ch_\beta \psi - \alpha^3\eta s_\alpha \psi ch_\beta \theta \\
&+ \beta\gamma K\alpha^2 c_\alpha \theta sh_\beta \psi + \beta^3\gamma K c_\alpha \theta sh_\beta \psi) = 0
\end{aligned}$$

where $\theta = \xi_1 - \xi_2$, $\psi = \xi_1 + \xi_2$, $c_\alpha = \cos\alpha$, $s_\alpha = \sin\alpha$, $ch_\beta = \cosh\beta$, and $sh_\beta = \sinh\beta$.

NUMERICAL SIMULATIONS

The natural frequencies are the roots of the characteristic equation and they are obtained using the method of bisection in MATLAB. The simulation is based on a beam with the following data from Ref. [9]: diameter d = 0.05 m flexural rigidity EI = 63476.1 N/m, mass per unit length m = 15.387 Kg/m, length of the beam L = 1.0 m, spring stiffness k = 190428 N/m, and the mass M_{d} = 3.0775 Kg.

In the first simulation, the mass attached to the beam M_{c} and the tension are set equal to zero so that the system is identical to that of Ref. [9]. The results of the five lowest natural frequencies are tabulated in Table 1, and show perfect agreement. The corresponding frequencies of the bare beam (simply supported beam with no in-span mass and spring mass system) are also tabulated. The results obtained by including the axial force (or tension) and in-span mass are compared with the finite element method and this is tabulated in Table 2. The results of the finite element solution corroborate those of the analytical solution. Figure 2 shows the lowest five mode shapes of the system. The form of the second, third, fourth, and fifth mode shapes, are similar to those of the bare beam, except that they are shifted so that the nth mode shape of the bare beam corresponds to the $n^{th} + 1$ of the loaded beam.

In the next simulation, the effect of varying the mass attached to the beam M_{c} while keeping mass suspended from the spring $M_{\mathrm{d}} = 3.0775$ Kg constant is investigated. The results are tabulated in Table 3. As expected, the natural frequencies decrease with increasing mass M_{c}. However, the degree of the change in the first mode is insignificant compared with that of the second, third, and fourth mode. The fifth natural frequency is identical to that of the bare beam because the mass-spring-mass

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TABLE 1. VALIDATION OF THE ANALYSIS FOR S = 0; $M_c = 0$; $\xi = 0.75$

Mode	Natural frequency (rad/s)		
	Present	Ref. [9]	Bare beam
1	243.858	243.858	633.900
2	645.203	645.203	2535.601
3	2540.530	2540.530	5705.101
4	5706.188	5706.188	10142.402
5	10142.400	10142.400	15847.503

TABLE 2. VALIDATION OF THE ANALYSIS FOR S = 1; $\xi = 0.75$; $r_m = 0.3$

Mode	Natural frequency (rad/s)	
	Present	Finite Element
1	248.352	248.352
2	554.950	554.950
3	2116.829	2116.842
4	4693.789	4693.939
5	8255.263	8256.074

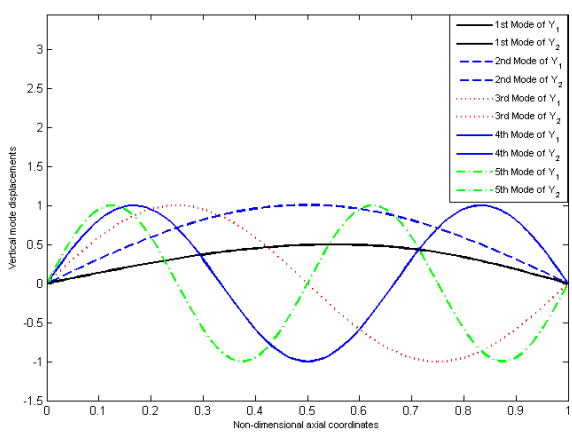

FIGURE 2. THE FIRST FIVE MODE SHAPE OF THE LOADED BEAM FOR S = 0 AND $M_c = 0$.

system is attached at its nodal point. Comparing the natural frequencies shown in the aforementioned table with those of the bare beam, it is seen that the natural frequencies of the loaded beam are lower than those of the bare beam. Note that the fundamental frequency of the loaded beam is very close to that of the mass-spring-mass system (ω = 248.6343 rad/s).

TABLE 3. EFFECT OF THE IN-SPAN MASS M_c ON THE NATURAL FREQUENCY FOR S = 0 AND $\xi = 0.75$.

M_c	Natural frequency (rad/s)				
	ω_1	ω_2	ω_3	ω_4	ω_5
0.923	243.802	626.828	2406.058	5567.392	10142.402
1.539	243.764	615.278	2333.338	5501.357	10142.402
3.078	243.667	558.631	2129.822	5388.585	10142.402
6.155	243.462	543.451	2016.723	5269.682	10142.402

The effect of varying the tension on the natural frequencies is show in Table 4. As expected the natural frequencies increase with increasing tension. The change in the natural frequencies of the first mode is less significant compared with that of the other four modes. This indicates that the mass-spring-mass system dominates the fundamental mode in this case.

TABLE 4. EFFECT OF THE TENSION ON THE NATURAL FREQUENCY FOR $M_d = 0$ AND $\xi = 0.75$.

S	Natural frequency (rad/s)				
	ω_1	ω_2	ω_3	ω_4	ω_5
0.2	243.878	646.456	2541.812	5707.473	10143.686
0.5	243.983	652.998	2548.531	5714.210	10150.427
0.8	244.164	664.981	2560.961	5726.701	10162.934
1.0	244.319	675.857	2572.381	5738.207	10174.465

The investigation of the effect of the location of the mass-spring-mass system is based on a ratio $\frac{M_c}{M_d} = 0.3$ and a nondimensional tension S=1. The results are tabulated in Table 5. All five natural frequencies changed with the location of the mass-spring- mass system.

In the following numerical simulation, the spring-mass system is tuned to the first natural frequency of the bare beam (M_d = 0.4169 kg, k = 190428.3 N/m); the tension, T = 63476.1 N, and the first five modes of the loaded beam are determined for various nondimensional locations ξ_1 of the mass-spring-mass system. The results are presented in Table 6. When the spring-mass system is located close to the end of the beam, the fundamental frequency is closest to that of the spring-mass system, i.e.

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TABLE 5. NATURAL FREQUENCIES OF THE LOADED BEAM FOR VARIOUS ξ (S = 1; r_m = 0.3).

ξ_1	Natural frequency (rad/s)				
	ω_1	ω_2	ω_3	ω_4	ω_5
0.1	247.712	663.508	2516.638	5523.882	9694.162
0.25	244.273	656.567	2436.240	5598.683	10174.465
0.5	240.874	648.860	2567.513	5449.169	10174.465

675.8492 rad/s. This is because the equivalent stiffness of the system tends to increase toward that of the spring when the mass-spring-mass is mounted near an end of the beam.

TABLE 6. NATURAL FREQUENCIES OF THE LOADED BEAM FOR VARIOUS ξ_1 WHEN THE SPRING MASS IS TUNED TO THE FIRST MODE OF THE BARE BEAM (M_c = 0; S = 1; M_d = 0.4169

ξ	Natural frequency (rad/s)				
	ω_1	ω_2	ω_3	ω_4	ω_5
0.1	695.371	2569.301	5738.556	10175.570	15880.365
0.25	616.399	727.746	2572.693	5738.22	10174.465
0.5	596.191	753.781	2567.513	5739.313	10174.465

In Fig. 3, where the spring-mass system is attached at ξ_1 = 0.1, only the fifth mode shape exhibits a different trend than those of the bare beam. The form of the other four mode shapes remain relatively similar. This implies that only the fifth mode is affected by the spring-mass system. For ξ = 0.25, Fig. 4 shows that the mode shapes exhibit trends similar to those observed in Fig. 2, except for the third mode which tends to be more steady. This perhaps is manifestation of the attachment point of the spring-mass system coinciding with an antinode.

The results in Fig. 5 indicate that when the spring-mass system is placed in the middle of the beam , the third and fifth mode shapes are similar to those of Fig. 2. The second mode shape is similar to that of the bare beam. This is due to the fact that the attachement point of the system coincides with the node of the second mode shape. On the other hand, the first and fourth mode shape are different than those from the previous figures. The first mode shifts to the second mode shape of the bare beam.

CONCLUSIONS

The derivation of the frequency equation of a simply supported beam subjected to an axial load with an attached mass-spring-mass is presented for the first time. This characteristic

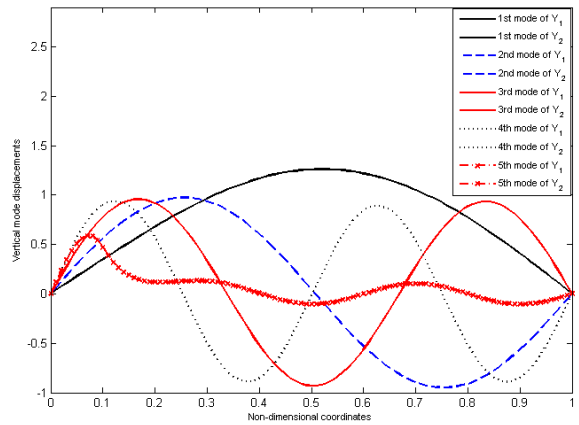

FIGURE 3. MODE SHAPES OF THE LOADED BEAM FOR S = 1, ξ = 0.1, AND M_c = 0.

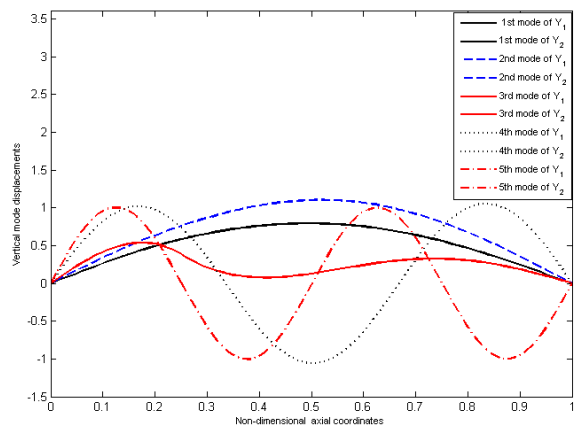

FIGURE 4. MODE SHAPES OF THE LOADED BEAM FOR S = 1, ξ = 0.25, AND M_c = 0.

equation is reduced to very simplified formula because of the two reference frames employed in the formulation. The validity of the formulation is examined via comparison with results in the literature and the finite element method. The results are shown to be in very good agreement.

The effect of the mass attached within the span of the beam, the tension (or axial load), and the location of the mass-spring-mass system are investigated. The results indicate that the fundamental frequency remains slightly constant as compared with the other four natural frequencies. The consequence of increasing the tension results in a significant increase in the natural frequencies of the loaded beam as expected. However, this is not the

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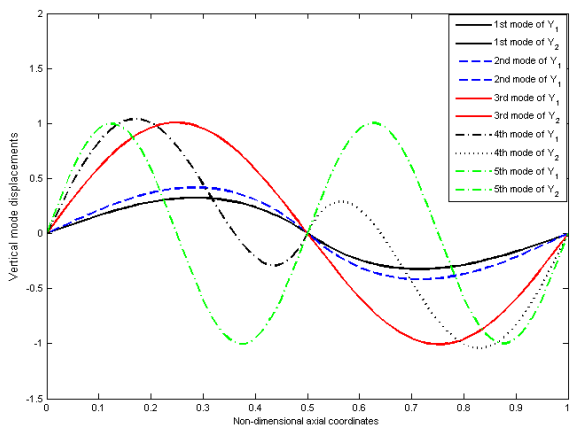

FIGURE 5. MODE SHAPES OF THE LOADED BEAM FOR S = 1, ξ = 0.5, AND M_c = 0.

case for the fundamental mode which remains very close to that of the mass-spring-mass system. This behavior of the fundamental frequency is observed when the mass attached to the beam is increased. Further, the first natural frequency of the loaded beam is generally much lower than that of the bare beam.

The location of the mass-spring-mass system shows the most effect on the modes. The natural frequencies can increase or decrease as the mass-spring-mass system is moved along the span of the beam.

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