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Technical Notes

Finite Element Free Vibration Analysis of Soft-Core Sandwich Beams

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Nomenclature

b	=	width of the beam
C_1, C_2, C_3, C_4	=	Lagrange cubic interpolation functions
E_c	=	Young's modulus of the core material
h, h_f	=	height of the beam, face sheet
h_b, h_c, h_p	=	one-half the height of the bottom, core, and top layer
L, L_e	=	length of the beam, a representative element
$\mathbf{N}_E, \mathbf{N}_T$	=	matrix of shape functions for face sheets modeled using Euler-Bernoulli or Timoshenko beam theory
$\mathbf{q}_{e_E}, \mathbf{q}_{e_T}$	=	vector of nodal displacements of a representative element for when the face sheets are modeled using Euler-Bernoulli or Timoshenko beam theory
$\mathbf{u}_E, \mathbf{u}_T$	=	vector of field variables for when the face sheets are modeled using Euler-Bernoulli or Timoshenko beam theory
$\bar{u}_b, \bar{u}_c, \bar{u}_p$	=	longitudinal displacement of a differential element of the bottom, core, and top layer
w_b, w_c, w_p	=	transverse displacement of the bottom, core, and top layer
z_b, z_c, z_p	=	local through the thickness of the bottom, core, and top layer
ν_c	=	Poisson's ratio of the core material
ρ_c, ρ_α	=	density of core layer, and that of the bottom or top layer
ϕ_b, ϕ_p	=	shear rotation of the bottom and top layer
ω	=	circular natural frequency
$\tilde{\omega}$	=	nondimensional circular natural frequency
$\dot{()}$	=	time derivative of ()
$()_i$	=	ith nodal component of ()
$()'$	=	partial derivative of () with respect to the longitudinal coordinate

Subscripts

b, c, p	=	bottom, core, and top layer
e	=	representative element
f	=	face sheet
α	=	layer bottom or top layer

Superscripts

e, (e)	=	representative element
T	=	transposition of the matrix

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I. Introduction

THIS paper revisits the problem of the free vibration of soft-core three-layer sandwich beams. An insight into the history of sandwich structures, the current trends, and future expectations can be gleaned from the paper by Vinson [1], who also provided arguments for continued research.

The traditional proposed kinematics (see [2,3], for example) do not permit core breathing or thickness stretching and cannot capture boundary conditions that are solely applied to the face sheets. Technological innovations in manufacturing and material science have increased the use of soft- or flexible-core materials in the construction of sandwich structures. This has necessitated the use of high-order theories or kinematics that permit core stretching and investigations in this regard include [4–14].

The objective of this technical note is twofold. The first is to present an alternative derivation of the quasi-two-dimensional formulation of Bekuit et al. [14]. Second is to demonstrate its applicability to systems with laminated face sheets, which was not reported in [14]. It is conjectured that the variation of the core transverse deformation is quadratic through the thickness while that of the axial deformation is cubic. The distinctions here are:

1) The transverse deformation of the core is described using the components at the face-core interfaces and the component at the midline.

2) The axial deformation components at the face-core interfaces are employed in conjunction with the components located at one-third of the core thickness away from the midline to represent the core axial deformation.

Bai and Sun [8] also described their core deformation with identical polynomial order in the thickness coordinate. However, they involved Poisson's ratio, as well as transverse normal and shear deformations of the core. The present formulation solely employs displacements at physical locations along the height of the sandwich beam.

The performance of the proposed kinematics is investigated by comparing the obtained simulation results with those in the literature and also with those obtained using the ANSYS commercial finite element code.

II. Mathematical Formulation

The schematic of the proposed sandwich beam elements models is given in Fig. 1; the Euler-Bernoulli beam theory is used to model the face sheets in Fig. 1a, and the Timoshenko beam theory is employed for the face sheets in Fig. 1b. Hereafter, the former shall be identified by the acronym Q2DE and the latter by Q2DT. The field variables of the face sheets are given as

$$\bar{u}_\alpha(x, z_\alpha, t) = u_\alpha(x, t) - z_\alpha\phi_\alpha(x, t), \quad \text{and}$$
$$\bar{w}_\alpha(x, z_\alpha, t) = w_\alpha(x, t) \quad \text{for } \alpha \in \{p, b\} \tag{1}$$

The implication is that the assumptions of the face sheet model correspond to those of Timoshenko beam theory when the variable $\phi_\alpha(x, t)$ is independent, and to the Euler-Bernoulli beam theory when $\phi_\alpha(x, t) = \frac{\partial w_\alpha(x,t)}{\partial x}$. The core axial deformation is assumed to vary cubically in the through-the-thickness coordinate, and the transverse deformation varies quadratically. These displacement field variables are written as

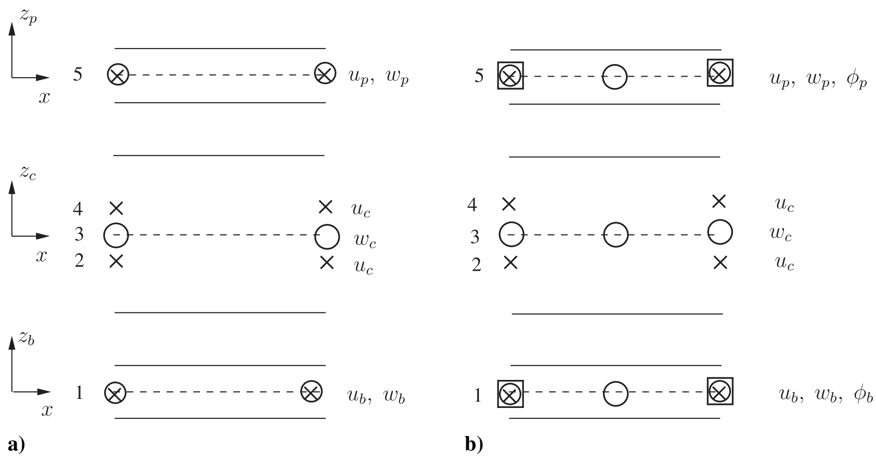

Fig. 1 Schematic of the proposed sandwich beam elements: (a) Q2DE and (b) Q2DT.

$$\bar{u}_c(x, z_c, t) = C_1[u_b(x,t) - h_b\phi_b(x,t)] + C_2 u_{c2} + C_3 u_{c3} + C_4[u_p(x,t) + h_p\phi_t(x,t)] \tag{2}$$

$$\bar{w}_c(x, z_c, t) = \frac{z_c}{2h_c}\left(1 + \frac{z_c}{h_c}\right)w_t(x,t) + \left[1 - \left(\frac{z_c}{h_c}\right)^2\right]w_c(x,t) - \frac{z_c}{2h_c}\left(1 - \frac{z_c}{h_c}\right)w_b(x,t) \tag{3}$$

where $u_{c2} = \bar{u}_c(x, z_c = -\frac{h_c}{3}, t)$, $u_{c3} = \bar{u}_c(x, z_c = \frac{h_c}{3}, t)$,

$$C_1 = -\frac{9}{16h_c^3}\left(z_c^3 - h_c z_c^2 - \frac{1}{9}h_c^2 z_c + \frac{1}{9}h_c^3\right)$$
$$C_2 = \frac{27}{16h_c^3}\left(z_c^3 - \frac{1}{3}h_c z_c^2 - h_c^2 z_c + \frac{1}{3}h_c^3\right)$$
$$C_3 = -\frac{27}{16h_c^3}\left(z_c^3 + \frac{1}{3}h_c z_c^2 - h_c^2 z_c - \frac{1}{3}h_c^3\right) \quad \text{and}$$
$$C_4 = \frac{9}{16h_c^3}\left(z_c^3 + h_c z_c^2 - \frac{1}{9}h_c^2 z_c - \frac{1}{9}h_c^3\right) \tag{4}$$

When the face sheets are modeled by Timoshenko beam theory, the axial deformation of each layer is interpolated linearly along the longitudinal axis, whereas the transverse deformation is interpolated quadratically.

The vector of field variables is written as

$$\mathbf{u}_T = [u_p \quad w_p \quad \phi_p \quad u_{c3} \quad u_{c2} \quad w_c \quad u_b \quad w_b \quad \phi_b]^T = \mathbf{N}_T\mathbf{q}_{e_T} \tag{5}$$

where the vector of nodal displacements is written as $\mathbf{q}_{e_T}^T = [u_{p_1} \quad w_{p_1} \quad \phi_{p_1} \quad u_{c31} \quad u_{c21} \quad w_{c_1} \quad u_{b_1} \quad w_{b_1} \quad \phi_{b_1} \quad w_{p_2} \quad w_{c_2} \quad w_{b_2} \quad u_{p_3} \quad w_{p_3} \quad \phi_{p_3} \quad u_{c33} \quad u_{c23} \quad w_{c_3} \quad u_{b_3} \quad w_{b_3} \quad \phi_{b_3}]$ The interpolation of the axial displacement variable of the Q2DE model is identical to that of Q2DT. However, the transverse displacement is interpolated using a Hermite cubic polynomial. The vector of field variables is given as

$$\mathbf{u}_E = [u_p \quad w_p \quad u_{c3} \quad u_{c2} \quad w_c \quad u_b \quad w_b]^T = \mathbf{N}_E\mathbf{q}_{e_E} \tag{6}$$

where the vector of nodal displacements is defined as $\mathbf{q}_{e_E}^T = [u_{p_1} \quad w_{p_1} \quad w'_{p_1} \quad u_{c31} \quad u_{c21} \quad w_{c_1} \quad w'_{c_1} \quad u_{b_1} \quad w_{b_1} \quad w'_{b_1} \quad u_{p_2} \quad w_{p_2} \quad w'_{p_2} \quad u_{c32} \quad u_{c22} \quad w_{c_2} \quad w'_{c_2} \quad u_{b_2} \quad w_{b_2} \quad w'_{b_2}]$. The global finite dimensional equations of motion of the system are readily derived following the procedure outlined in [14].

Table 1 Material and geometric properties for simulation #1

Component	Material	E (GPa)	ν	ρ (kg/m^3)
Face sheets	Steel	210	0.30	7900
Core	Divinycell® H60	0.056	0.27	60
$L = 260$ mm; $w = 59.9$ mm; $h_f = 1.9$ mm; $2h_c = 34.8$ mm				

III. Discussion of Results

The material and geometric properties of the partially-cantilevered sandwich beam (i.e., only the face sheets are cantilevered) used in the first simulation, which are tabulated in Table 1, are obtained from [6,7]. Table 2 shows the first ten natural frequencies. The results in the last and fourth columns are obtained using ANSYS commercial finite element software models in which Plane82 elements are employed. Specifically, 1200 Plane82 elements (150 in the axial

Table 2 Comparison of natural frequencies for partially-cantilevered sandwich beam (simulation #1); see also Table 1 of Sokolinksy and Nutt [7]

	Frequency (Hz)									
Mode	Exp. [6]	Classical	FE [6]	Frostig and Baruch [5]	Sokolinksy and Nutt [7]	Moreira and Dias Rodriguez [11]	Q2DSB	Q2DT	Q2DE	FE
1		152	165	165	165	165	164	164	164	165
2	544	476	512	511	512	512	508	508	508	512
3	950	859	913	910	912	912	902	902	903	913
4	1391	1316	1379	1373	1378	1376	1357	1357	1357	1380
5	1954	1871	1939	1928	1940		1897	1897	1899	1941
6		2532	2476	2393	2392	2578	2485	2485	2485	2472
7	2350–2400	3311	2509	2398	2395	2562	2510	2510	2510	2505
8		4208	2558	2430	2425	2617	2539	2539	2542	2551
9		4799	2567	2534	2524	2727	2559	2559	2559	2564
10	2511	5074	2608	2590	2612		2564	2564	2564	2610

Table 3 Material and geometric properties for simulation #2

Component	Material properties
Face sheets	$E_{11} = 131$ GPa, $E_{22} = E_{33} = 10.34$ GPa, $\nu_{12} = \nu_{13} = 0.22$ $\nu_{23} = 0.49$, $G_{12} = G_{23} = 6.895$ GPa, $G_{13} = 6.205$ GPa, $\rho = 1627$ kg/m^3
Isotropic core	$E = 6.89$ MPa, $G = 3.45$ MPa, $\rho = 97$ kg/m^3
$L = 300$ mm and 120 mm; $w = 20$ mm; $h_f = 0.25$ mm; $2h_c = 25$ mm	

Table 4 Comparison of natural frequencies for sandwich beam with simply supported laminated face sheets (simulation #2); see also Table 10 of Vidal and Polit [10]

		Nondimensional frequency $\tilde{\omega}$				
$\frac{L}{h}$	Model	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
4	GLHT[a]	0.638	1.800	3.631	6.133	9.276
	ZZT[a]	0.616	1.772	3.621	6.186	9.464
	HSDT-33[a]	1.495	3.277	5.497	7.735	10.182
	HSDT-Reddy[a]	2.249	4.682	7.293	10.176	13.396
	SinRef-7p[a]	0.645	1.819	3.674	6.223	9.463
	Q2DT	0.674	1.716	2.175	2.643	3.645
	Q2DE	0.674	1.722	2.176	2.646	3.675
	FE	0.580	1.728	2.144	2.641	3.510
10	GLHT[a]	1.346	2.993	5.126	7.860	11.250
	ZZT[a]	1.282	2.874	4.970	7.685	11.075
	HSDT-33[a]	3.468	7.355	11.402	15.748	20.481
	HSDT-Reddy[a]	5.012	11.075	17.051	23.089	29.263
	SinRef-7p[a]	1.361	3.026	5.183	7.961	11.442
	Q2DT	1.298	2.794	4.982	7.548	11.059
	Q2DE	1.299	2.795	4.986	7.560	11.094
	FE	1.242	2.795	4.864	7.544	10.924

[a]Taken from Vidal and Polit .

direction, 4 through the core, and 2 each through the face sheets) are used for the last column. The poor performance of the classical formulation method for situations of very low ratio of the core elastic modulus to the face sheet elastic modulus (2.6667×10^{-4} in this case) is readily observable. Further, the classical formulation cannot capture the partially cantilevered boundary conditions. This formulation is based on the following assumptions:

1) The top and bottom layers are modeled as Euler-Bernoulli beams.

2) The layers have identical transverse displacement that is independent of the thickness coordinate.

3) The core is modeled as a Timoshenko beam.

The results of [5,7] are virtually identical, indicating the negligible effect of the definition of the acceleration of the transverse displacement field variable, which was modified in the latter. Only the three-layer layw4x results of Moreira and Dias Rodrigues [11] are reproduced in Table 2 (i.e., column seven) because they are better than those with five-layer layw4x. These results deteriorate between modes 6 and 9, where core stretching is observed. The results using the original quasi-two-dimensional formulation (Q2DSB) of Beikut et al. [14] are in excellent agreement with those of Q2DT and Q2DE; 200 elements are used in each formulation. Although the quasi-two-dimensional results are the best during core stretching, they are not as good in the preceding lower modes, especially modes 4 and 5.

For a system with face sheets that are made from laminated unidirectional composites, the material properties are taken from Vidal and Polit [10] and are tabulated in Table 3. The finite element simulations using ANSYS modeled the core using SOLID95 elements and the face sheets by Shell99 layered elements. The system is unsymmetric with a 0/90/core/0/90 stacking sequence. For $\frac{L}{h} = 10$, 60 SOLID95 elements are used along the length of the core, 4 along the width, and 5 through the height. The number of SOLID95 elements employed along the length of the core for $\frac{L}{h} = 4$ is 24. The first five nondimensional natural frequencies, $\tilde{\omega}(= \frac{\omega L^2}{h}\sqrt{(\frac{\rho}{E_{22}})_f})$, for the case of simply supported boundary conditions of the face sheets and length-to-height ratios $\frac{L}{h} = 4$ and 10, are tabulated in Table 4. The HSDT-33 is a high-order theory that uses third-order expansions for the displacement fields; the ZZT is a zig-zag theory that satisfies interface transverse shear continuity; the GLHT is a global-local high-order theory; the HSDT-Reddy is Reddy high-order shear displacement theory; and the SinRef-7p is a seven-parameter sinus model in which the negative value of the slope of the midline transverse deformation is not assumed to be the coefficient of the linear component of the core through-the-thickness coordinate variable. The performances of the GLHT, ZZT, SinRef-7p, Q2DT, and Q2DE models, when benchmarked against the ANSYS commercial finite element software results, FE, are competitive. The Q2DT and Q2DE models capture modes 3 and 4, which exhibited noticeable core through-the-thickness deformation for $\frac{L}{h} = 4$.

IV. Conclusion

An alternative derivation of the quasi-two-dimensional sandwich element formulation of Bekuit et al. [14] for free-vibration analyses of soft-core three-layer sandwich beams is presented. Both yield identical results as expected. The application of the formulation to systems with laminated face sheets is demonstrated. Its performance when core stretching is absent is generally not as good as those involving core stretching.

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References

[1] Vinson, J. R., "Sandwich Structures: Past, Present, and Future," *Sandwich Structures 7: Advancing with Sandwich Structures and Materials*, edited by O. T. Thomsen, E. Bozhevolnaya, and A. Lyekegaard, Springer, Dordrecht, 2005, pp. 3–12.

[2] Mead, D. J., and Markus, S., "The Forced Vibration of a Three-Layer, Damped Sandwich Beam with Arbitrary Boudary Conditions," *Journal of Sound and Vibration*, Vol. 10, No. 2, 1969, pp. 163–175. doi:10.1016/0022-460X(69)90193-X

[3] Galucio, A. C., Deü, J.-F., and Ohayon, R., "Finite Element Formulation of Viscoelastic Sandwich Beams Using Fractional Derivative Operators," *Computational Mechanics*, Vol. 33, No. 4, 2004, pp. 282–291. doi:10.1007/s00466-003-0529-x

[4] Ojalvo, I. U., "Departures from Classical Beam Theory in Laminated Sandwich and Short Beams," *AIAA, Journal*, Vol. 15, No. 10, 1977, pp. 1518–1521. doi:10.2514/3.7449

[5] Frostig, Y., and Baruch, M., "Free Vibrations of Sandwich Beams with a Traversely Flexible Core: A High Order Approach," *Journal of Sound and Vibration*, Vol. 176, No. 2, 1994, pp. 195–208. doi:10.1006/jsvi.1994.1368

[6] Sokolinsky, V. S., von Bremen, H. F., Lavoie, J. A., and Nutt, S. R., "Analytical and Experimental Study of Free Vibration Response of Soft-Core Sandwich Beams," *Journal of Sandwich Structures & Materials*, Vol. 6, No. 3, 2004, pp. 239–261. doi:10.1177/1099636204034634

[7] Sokolinsky, V. S., and Nutt, S. R., "Consistent Higher-Order Dynamic Equations for Soft-Core Sandwich Beams," *AIAA Journal*, Vol. 42, No. 2, 2004, pp. 374–382. doi:10.2514/1.2742

[8] Bai, J. M., and Sun, C. T., "The Effect of Viscoelastic Adhesive Layers on Structural Damping of Sandwich Beams," *Mechanics of Structures and Machines*, Vol. 23, No. 1, 1995, pp. 1–16. doi:10.1080/08905459508905225

[9] Amirani, M. C., Khalili, S. M. R., and Nemati, N., "Free Vibration Analysis of Sandwich Beam with FG Core Using the Element Free Galerkin Method," *Composite Structures*, Vol. 90, No. 3, 2009, pp. 373–379. doi:10.1016/j.compstruct.2009.03.023

[10] Vidal, P., and Polit, O., "Vibration of Multilayered Beams Using Sinus Finite Elements with Transverse Normal Stress," *Composite Structures*, Vol. 92, No. 6, 2010, pp. 1524–1534. doi:10.1016/j.compstruct.2009.10.009

[11] Moreira, R. A. S., and Dias Rodrigues, J., "Static and Dynamic Analysis of Soft Core Sandwich Panels with Through-Thickness Deformation," *Composite Structures*, Vol. 92, No. 2, 2010, pp. 201–215. doi:10.1016/j.compstruct.2009.07.015

[12] Oskooei, S., Hansen, J. S., "Higher-Order Finite Element for Sandwich Plates," *AIAA, Journal*, Vol. 38, No. 3, 2000, pp. 525–533. doi:10.2514/2.991

[13] Nabarrete, A., de Almeida S. F. M., and Hansen, J. S., "Sandwich-Plate Vibration Analysis: Three-Layer Quasi-Three-Dimensional Finite Element Model," *AIAA, Journal*, Vol. 41, No. 8, 2003, pp. 1547–1555. doi:10.2514/2.2106

[14] Bekuit, J.-J. R. B., Oguamanam, D. C. D., and Damisa, O., "A Quasi-2D Finite Element Formulation for the Analysis of Sandwich Beams," *Finite Elements in Analysis and Design*, Vol. 43, No. 14, 2007, pp. 1099–1107. doi:10.1016/j.finel.2007.08.005

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