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NONLINEAR VIBRATIONS OF A BEAM- SPRING-LARGE MASS SYSTEM

Mohammad A. Bukhari
School of Science and Engineering
Central Michigan University
Mt. Pleasant, Michigan 48859
Email: bukha2m@cmich.edu

Oumar R. Barry*
School of Science and Engineering
Central Michigan University
Mt. Pleasant, Michigan 48859
Email: barry1o@cmich.edu

ABSTRACT

This paper presents the nonlinear vibration of a simply supported Euler-Bernoulli beam with a mass-spring system subjected to a primary resonance excitation. The nonlinearity is due to the mid-plane stretching and cubic spring stiffness. The equations of motion and the boundary conditions are derived using Hamiltons principle. The nonlinear system of equations are solved using the method of multiple scales. Explicit expressions are obtained for the mode shapes, natural frequencies, nonlinear frequencies, and frequency response curves. The validity of the results is demonstrated via comparison with results in the literature. Exact natural frequencies are obtained for different locations, rotational inertias, and masses.

INTRODUCTION

Since beams are used in many engineering models, studying the nonlinear vibration of beams has received a considerable attention. This nonlinearity may be attributable to geometric, inertial, or material in nature. When the beam has immovable end conditions, we encounter geometric nonlinearity in the modeling called streching nonlinerarity.

Studying the nonlinear vibrations of beams has been reviewed up to 1979 by Nayfeh and Mook [1]. Approximate analytical solutions for these types of problems can be obtained using perturbation methods. These perturbation techniques have been reviewed by Nayfeh [2]. Following that, the nonlinear free vibrations of a beam with spring-mass system have been investigated by Dowell [3]. The nonlinearity in Dowell's study was due to cubic nonlinearity stemming from spring constant. However, nonlinearity due to the mid-plane stretching was not examined . Dowell's work has been extended by Pakdemirli and Nayfeh [4] by including the effect of stretching, damping, and primary resonance excitation. Furthermore, Barry et al. [5] have extended Pakdemirli and Nayfeh work by considering the effect of axial tension, attached system damping, and multi attached systems. More works on the nonlinear vibrations of beam with mass or multi-masses and different boundary conditions can be found in [6]- [9].

Numerous studies have been reported in the literature about the nonlinear vibrations of beams; however, no work has been reported about the effect of rotational inertia of the attached system. In the current study, we extend Pakdemirli and Nayfeh work by considering the rotational inertia. The multiple scales method, a perturbation technique, is used to obtain an approximate analytical solutions for the nonlinear equations of motion and boundary conditions. Parametric studies are conducted to examine the effect of rotational inertia on the linear frequencies, nonlinear frequencies, and frequency response curves.

MATHEMATICAL MODEL

A schematic diagram for the system is shown in Fig.1. The considered system is a simply supported Euler-Bernoulli beam with a spring-mass system located at $x = x_{s1}$ from the left reference frame. Following [1], the equations of motion and the

*Corresponding author: O. Barry, barry1o@cmich.edu

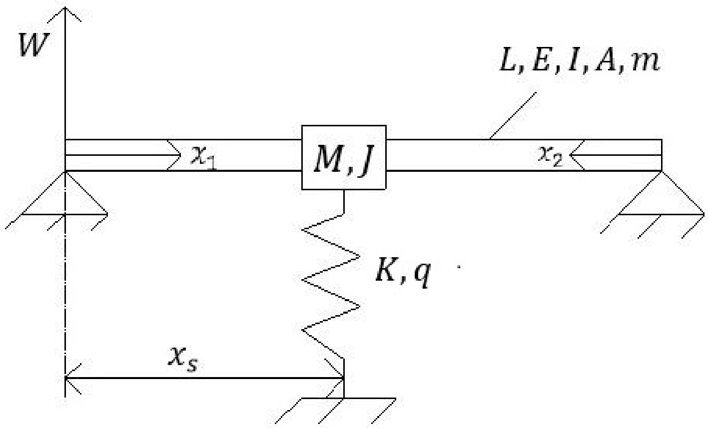

FIGURE 1. A schematic diagram for the system.

boundary conditions for the system can be obtained as

$$m\ddot{W}_i + EIW_i^{iv} = \frac{EA}{2L}[\sum_{r=1}^{2}\int_0^{x_{sr}} W_r'^2 dx_r]W_i'' \quad for \quad i = 1,2 \tag{1}$$

$$W_i(0,t) = W_i''(0,t) = 0 \tag{2}$$

$$W_1(x_{s1},t) = W_2(x_{s2},t) \tag{3}$$

$$W_1'(x_{s1},t) = -W_2'(x_{s2},t) \tag{4}$$

$$W_1''(x_{s1},t) = W_2''(x_{s2},t) - J\ddot{W}_1'(x_{s1},t) \tag{5}$$

$$EI[W_1'''(x_{s1},t) + W_2'''(x_{s2},t)] = KW_1(x_{s1},t) + qW_1^3(x_{s1},t) + M\ddot{w}_p(x_{s1},t) \tag{6}$$

where L is the length of the beam, m represents the mass per unit length of the beam, EI and EA are the flexural and the axial rigidity, respectively, M is the total mass of spring mass system, K and q are the linear and nonlinear spring constant, respectively, W is the transverse displacement, x_i is the axial coordinate, and t is the time. The primes represents the derivative with respect to axial coordinate and the dots denotes the derivative with respect to time. The subscript 1 and 2 refers to the left and right beam respectively.

Next, we introduce the following dimensionless parameters

$$\xi_i = \frac{x}{L}; \xi_{si} = \frac{x_{si}}{L}; w_i = \frac{W_i}{r}; \tau = \frac{t}{l^2}\sqrt{\frac{EI}{m}}; \alpha = \frac{M}{mL}; k = \frac{KL^3}{EI}; \gamma = \frac{qL^3r^2}{EI}; \eta = \frac{J}{mL^3}; \tag{7}$$

where r is the radius of gyration.

Introducing these parameters in Eqs. (1) - (6) leads to

$$\ddot{w}_i + w_i^{iv} = \frac{1}{2}[\sum_{r=1}^{2}\int_0^{\xi_{sr}} w_r'^2 d\zeta_r]w_i'' \tag{8}$$

$$w_i(0,\tau) = w_i''(0,\tau) = 0 \tag{9}$$

$$w_1(\xi_{s1},\tau) = w_2(\xi_{s2},\tau) \tag{10}$$

$$w_1'(\xi_{s1},\tau) = -w_2'(\xi_{s2},\tau) \tag{11}$$

$$w_1''(\xi_{s1},\tau) = w_2''(\xi_{s2},\tau) - \eta\ddot{w}_1'(\xi_{s2},\tau) \tag{12}$$

$$w_1'''(\xi_{s1},\tau) + w_2'''(\xi_{s2},\tau) = kw_1(\xi_{s1},\tau) + \gamma w_1^3(\xi_{s1},\tau) + \alpha\ddot{w}_1(\xi_{s1},\tau) \tag{13}$$

Adding damping and forcing terms to Eq. (8) yields

$$\ddot{w}_i + w_i^{iv} = \frac{1}{2}[\sum_{r=1}^{2}\int_0^{\xi_{sr}} w_r'^2 d\zeta_r]w_i'' - 2\bar{\mu}\dot{w}_i + \bar{F}_i\cos\Omega t \tag{14}$$

where $\bar{\mu}$, $\bar{F}_i$, and Ω are the dimensionless internal damping coefficient, the dimensionless excitation amplitude, and the dimensionless excitation frequency, respectively.

By employing the method of multiple scales, the expansion of beam's displacement can be represented as

$$w_i(\xi_i,\tau,\varepsilon) = \varepsilon w_{i1}(\xi_i,T_0,T_2) + \varepsilon^3 w_{i3}(\xi_i,T_0,T_2) + ... \tag{15}$$

where $T_0 = \tau$ and $T_2 = \varepsilon^2\tau$. Since ε is a small dimensionless parameter, T_0 is a fast time scale and T_2 is a slow time scale. The scale T_1 is missing because the type of nonlinearity here is cubic. In order to get the effect of force and damping in the same perturbation equation, we assume $\bar{\mu} = \varepsilon^2\mu$, $\bar{v} = \varepsilon^2 v$ and $\bar{F} = \varepsilon^2 F_i$ for primary resonance case.

The derivatives with respect to nondimentional time become expansions in terms of the partial derivative with respect to T_n as

$$(\cdot) = D_0 + \varepsilon^2 D_2 \tag{16}$$

$$(\cdot\cdot) = D_0^2 + 2\varepsilon^2 D_0 D_2 \tag{17}$$

where $D_n = \frac{\partial}{\partial T_n}$.

Substituting Eqs. (15) - (17) into the nondimentional equations of motion and boundary conditions and separating the terms of similar power leads to

order ε

$$D_0^2 w_{i1} + w_{i1}^{iv} = 0 \tag{18}$$

$$w_{i1}(0,\tau) = w_{i1}''(0,\tau) = 0 \tag{19}$$

$$w_{11}(\xi_{s1},\tau) = w_{21}(\xi_{s2},\tau) \tag{20}$$

$$w_{11}'(\xi_{s1},\tau) = -w_{21}'(\xi_{s2},\tau) \tag{21}$$

$$w_{11}''(\xi_{s1},\tau) = w_{21}''(\xi_{s2},\tau) - \eta D_0^2 w_{11}'(\xi_{s1},\tau) \tag{22}$$

$$w_{11}'''(\xi_{s1},\tau) + w_{21}'''(\xi_{s2},\tau) = k w_{11}(\xi_{s1},\tau) + \alpha D_0^2 w_{11}(\xi_{s1},\tau) \tag{23}$$

order ε^3

$$\begin{aligned} D_0^2 w_{i3} + w_{i3}^{iv} = \frac{1}{2}[\sum_{r=1}^{2}\int_0^{\xi_{sr}} w_{r1}'^2 d\zeta_r] w_{i1}'' - 2D_0 D_2 w_{i1} \\ -2\mu D_0 w_{11} + F_i \cos\Omega T_0 \end{aligned} \tag{24}$$

$$w_{i3}(0,\tau) = w_{i3}''(0,\tau) = 0 \tag{25}$$

$$w_{13}(\xi_{s1},\tau) = w_{23}(\xi_{s2},\tau) \tag{26}$$

$$w_{13}'(\xi_{s1},\tau) = -w_{23}'(\xi_{s2},\tau) \tag{27}$$

$$\begin{aligned} w_{13}''(\xi_{s1},\tau) = w_{23}''(\xi_{s2},\tau) \\ -\eta[D_0^2 w_{13}'(\xi_{s1},\tau) + 2D_0 D_2 w_{11}'(\xi_{s1},\tau)] \end{aligned} \tag{28}$$

$$\begin{aligned} w_{13}'''(\xi_{s1},\tau) + w_{23}'''(\xi_{s2},\tau) = k w_{13}(\xi_{s1},\tau) + \gamma w_{11}^3(\xi_{s1},\tau) \\ +\alpha[D_0^2 w_{13}(\xi_{s1},\tau) + 2D_0 D_2 w_{11}(\xi_{s1},\tau)] \end{aligned} \tag{29}$$

LINEAR PROBLEM

The solution of the problem at order ε can be obtained linearly. Therefore the solution can be assumed as

$$w_{i1} = [A_1(T_2)e^{j\omega T_0} + cc]Y_i(\xi_i) \tag{30}$$

where *cc* is the complex conjugate for the preceding terms. Introducing Eq. (30) into Eqs. (18)-(23) leads to

$$Y_i^{iv} - \omega^2 Y_i = 0 \tag{31}$$

$$Y_i(0) = Y_i''(0) = 0 \tag{32}$$

$$Y_1(\xi_{s1}) = Y_2(\xi_{s2}) \tag{33}$$

$$Y_1'(\xi_{s1}) = -Y_2'(\xi_{s2}) \tag{34}$$

$$Y_1''(\xi_1) = Y_2''(\xi_2) + \omega^2 \eta Y_1'(\xi_1) \tag{35}$$

$$Y_1'''(\xi_{s1}) + Y_2'''(\xi_{s2}) = kY_1(\xi_{s1}) - \omega^2 \alpha Y_1(\xi_1) \tag{36}$$

The mode shapes of the beams Y_i can be expressed as

$$Y_i(\xi_i) = c_{1i}\sin\beta\xi_i + c_{3i}\sinh\beta\xi_i \tag{37}$$

$$\beta = \sqrt{\omega} \tag{38}$$

where c_{1i} and c_{2i} are arbitrary constants. These constants and the natural frequencies of the system can be obtained by substituting Eq. (37) into Eqs. (33)-(36).One should note that the terms of c_{2i} and c_{4i} are missing because we use two different reference frame at each end of the boundary conditions in Eq. (32).

NONLINEAR PROBLEM

The problem at order ε^3 is nonlinear. The inhomogeneous solution of Eqs. (24) - (29) has secular terms. These secular terms must be vanished in order to make the expansion in Eq. (15) uniformly valid as τ increases. To achieve that the solvability condition must be satisfied [2]. The solvability condition can be obtained by representing the solution as

$$w_{i3} = \phi_i(\xi_i, T_2)e^{j\omega T_0} + cc + W_i^*(\xi_i, T_0, T_2) \tag{39}$$

where W_i^* is unique and free of small-divisor terms and secular terms. When we have a primary resonance case, the excitation frequency detunes from one of the natural frequency as

$$\Omega = \omega + \varepsilon^2\sigma \tag{40}$$

where σ is a detuning parameter. Introducing the expressions in Eq.(30) and Eqs. (39) - (40) into Eqs. (24) - (29), and collecting the coefficient of secular terms in order to delete them yields

$$\begin{aligned} \phi_i^{iv} - \omega^2\phi_i = \frac{3}{2}\bar{A_1}A_1^2[\sum_{r=1}^{2}\int_0^{\xi_{sr}} Y_r'^2(\xi_r)d\zeta_r]Y_i''(\xi_s i) \\ -2j\omega(A_1' + \mu A_1)Y_i(\xi_{si}) + \frac{1}{2}F_i e^{j\sigma T_2} \end{aligned} \tag{41}$$

$$\phi_i(0, T_2) = \phi_i''(0, T_2) = 0 \tag{42}$$

$$\phi_1(\xi_{s1}, T_2) = \phi_2(\xi_{s2}, T_2) \tag{43}$$

$$\phi_1'(\xi_{s1}, T_2) = -\phi_2'(\xi_{s2}, T_2) \tag{44}$$

$$\begin{aligned} \phi_1''(\xi_{s1}, T_2) = \phi_2''(\xi_{s2}, T_2) \\ -\eta[-\omega^2\phi_1'(\xi_{s1}, T_2) + 2j\omega A_1'Y_1'(\xi_1)] \end{aligned} \tag{45}$$

$$\begin{aligned} \phi_1'''(\xi_{s1}, T_2) + \phi_2'''(\xi_{s2}, T_2) = k\phi_1(\xi_{s1}, T_2) \\ +3\gamma A_1^2\bar{A_1}Y_1^3(\xi_{s1}) + \alpha[-\omega^2\phi_1(\xi_{s1}, T_2) + 2j\omega A_1'Y_1(\xi_1)] \end{aligned} \tag{46}$$

Rearranging Eqs. (41) - (46) leads to the following solvability condition

$$\begin{aligned} 2j\omega(A_1' + \mu A)b_1 + 3\bar{A_1}A_1^2(\frac{b_2^2}{2} + \gamma Y_1^4) - \frac{1}{2}fe^{j\sigma T_2} \\ +j\omega[2\alpha A_1'Y_1^2(\xi_{s1}) + 2\eta A_1'Y_1'^2(\xi_{s1})] = 0 \end{aligned} \tag{47}$$

where

$$b_1 = \sum_{r=1}^{2}\int_0^{\xi_{sr}} Y_r^2 d\xi_r \tag{48}$$

$$b_2 = \sum_{r=1}^{2}\int_0^{\xi_{sr}} Y_r'^2 d\xi_r \tag{49}$$

$$b_3 = \sum_{r=1}^{2}\int_0^{\xi_{sr}} Y_r''Y_r d\xi_r \tag{50}$$

$$f = \sum_{r=1}^{2}\int_0^{\xi_{sr}} F_rY_r d\xi_r \tag{51}$$

We note that integrating b_3 yields $b_3 = -b_2$. Next, we Introduce the polar form as

$$A_1(T_2) = \frac{1}{2}a(T_2)e^{\theta(T_2)} \tag{52}$$

and the following expression in order to convert Eq. (47) into an autonomous form

$$\gamma_1 = \sigma T_2 - \theta \tag{53}$$

By substituting Eqs. (52) - (53) into the solvability condition, the following equations can be obtained after separating the real and imaginary components

$$\omega a' b_4 = \frac{1}{2} f \sin \gamma_1 - \omega b_5 a \tag{54}$$

$$\omega a(\sigma - \gamma_1') b_4 = -\frac{1}{2} f \cos \gamma_1 + \frac{3}{8} a^3 b_6 \tag{55}$$

where

$$b_4 = b_1 + \alpha Y_1^2(\xi_{s1}) + \eta Y_1'^2(\xi_{s1}) \tag{56}$$

$$b_5 = b1\mu \tag{57}$$

$$b_6 = \frac{b_2^2}{2} + \gamma Y_1^4(\xi_{s1}) \tag{58}$$

The nonlinear frequencies can be obtained by making $\sigma = f = \mu = 0$, then one can obtain from Eqs. (54) - (55) the following

$$a' = 0 \Rightarrow a = constant \tag{59}$$

$$\omega a b_4 \gamma_1' = -\frac{3}{8} b_6 a^3 \tag{60}$$

Therefore the nonlinear frequency can be expressed as

$$\omega_{nl} = \omega + \frac{3 b_6 a^2}{8 \omega b_4} \tag{61}$$

To find frequency response,for periodic motion $a' = \gamma_1' = 0$, then , after elimenating γ_1, one can determine from Eqs. (54) and (55) the detuning parameter as

$$\sigma = \frac{3a^2 b_6}{8\omega b_4} \pm \sqrt{\frac{f^2}{4a^2\omega^2 b_4} - \frac{b_5^2}{b_4^2}} \tag{62}$$

NUMERICAL SIMULATION

Simulation in the present study has been carried out using Matlab. Some of the results are presented in order to show the effect of rotational inertia on the frequencies and frequency response curves. In all numerical calculation we use $k = \gamma = 2\pi^4$ [4].

TABLE 1. Validating the natural frequencies of a beam with spring-mass system for $\xi_{s1} = 0.1, \alpha = 0.5, k_1 = \gamma = 2\pi^4$

Data	1^{st} mode	2^{nd} mode	3^{rd} mode	4^{th} mode	5^{th} mode
Present	10.9850	35.2410	72.1800	131.6660	216.6070
Ref [4]	10.9844	35.2402	72.1795	131.6656	216.6067

TABLE 2. The lowest five natural frequencies of the system for various locations and rotational inertia

ξ_{s1}	η	1^{st} mode	2^{nd} mode	3^{rd} mode	4^{th} mode	5^{th} mode
0.1	0	10.985	35.241	72.18	131.666	216.607
	0.1	16.462	52.404	75.749	133.421	224.002
	0.25	17.807	54.668	76.031	133.452	224.044
0.3	0	14.297	33.241	87.176	146.198	213.798
	0.1	16.138	35.721	105.335	180.092	218.226
	0.25	16.679	35.938	105.526	180.339	218.268

First, in order to check the current model, we verify some of the results with those obtained in the literature. The comparisons between the natural frequencies are listed in Table.1 and show very good agreement.

The lowest five natural frequencies are tabulated in Table. 2 for different values of η and ξ_{s1}. It is observed that increasing the inertia increases the natural frequencies; however, this increasing does not have a significant effect on the higher modes.

The nonlinear frequency curves for various values of α and η are drown in Figs. 2-5. In Figs. 2-3, it is demonstrated that including the inertia increases both the linear and nonlinear frequencies. Moreover, this also mitigates the bending of curves. Increasing the total mass of the spring mass system reduces the linear and nonlinear frequencies as shown in Figs. 4-5.

The frequency response curves for different values of α and η are plotted in Figs. 6-7. The solid line represents the stable solutions, where as the dotted refers to the unstable solutions. It is revealed that increasing the rotational inertia alleviates the maximum vibration amplitude. However, increasing the total mass of the spring-mass system increases the maximum vibration amplitude while the reduction due to increasing the rotational inertia becomes more significant.

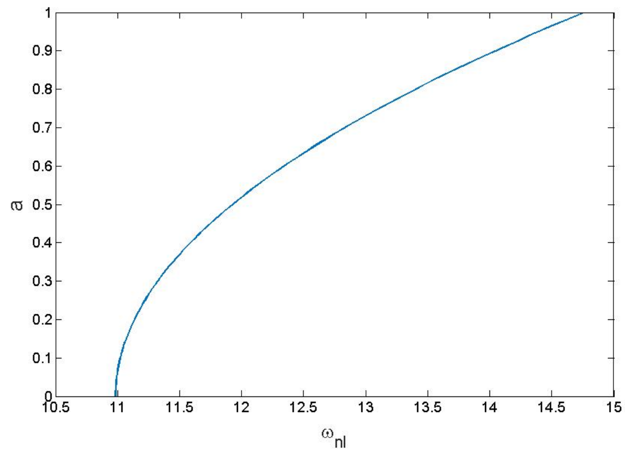

FIGURE 2. Nonlinear frequency vs. vibration amplitude: $k = \gamma = 2\pi^4$, $\alpha = 0.5$, $\eta = 0$, $\xi_{s1} = 0.1$ first mode of vibration.

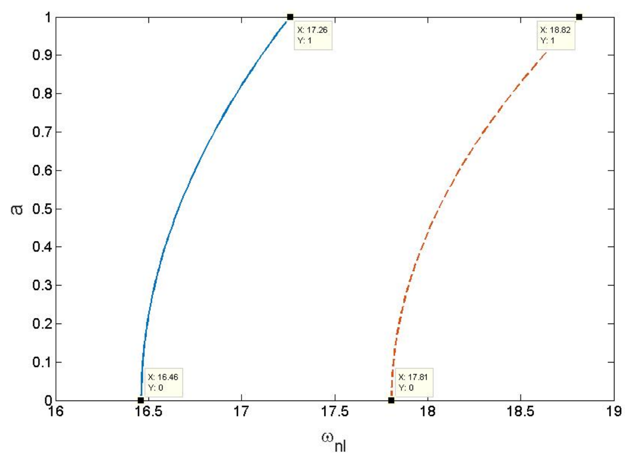

FIGURE 3. Nonlinear frequency vs. vibration amplitude: $k = \gamma = 2\pi^4$, $\alpha = 0.5$, $\xi_{s1} = 0.1$ first mode of vibration.

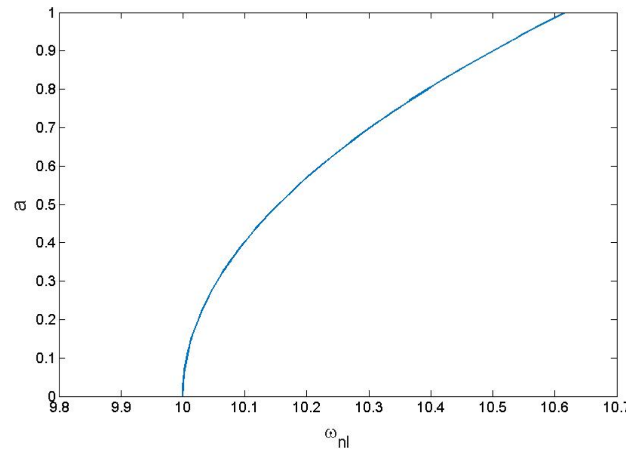

FIGURE 4. Nonlinear frequency vs. vibration amplitude: $k = \gamma = 2\pi^4$, $\alpha = 5$, $\eta = 0$, $\xi_{s1} = 0.1$ first mode of vibration.

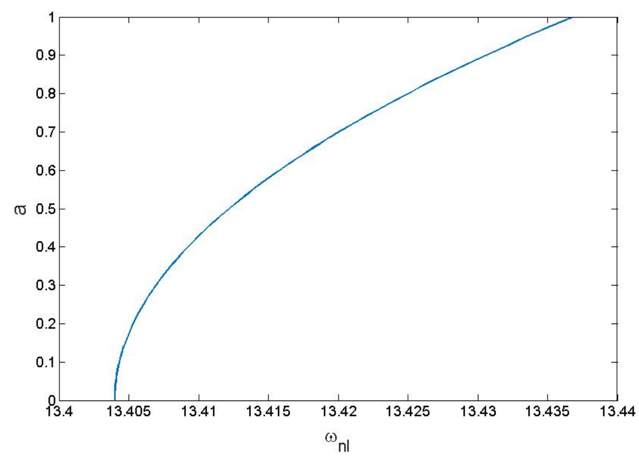

FIGURE 5. Nonlinear frequency vs. vibration amplitude: $k = \gamma = 2\pi^4$, $\alpha = 5$, $\eta = 0.1$, $\xi_{s1} = 0.1$ first mode of vibration.

CONCLUSION

In the present study, the nonlinear vibrations of a beam with spring-large mass system is investigated. The nonlinearity of the system stems from mid-plane stretching and cubic spring constant. The nonlinear problem is solved by the method of multiple scales to obtain an approximate analytical solution. Explicit expressions for the nonlinear frequency and detuning parameter are presented. The results are validated via comparisons with those in the literature. The effect of rotational inertia on the linear frequencies, the nonlinear frequencies, and the detuning parameter are examined. The numerical simulation reveals that the linear natural frequencies increase with inceasing rotational inertia . It is also demonstrated that the nonlinear frequencies increases as the inertia increases. Moreover, Increasing the inertia alleviates the effect of nonlinearity. Numerical results also indicate that maximum vibration amplitude increases with increasing total mass of the spring-mass system increases and decreasing rotational inertia.

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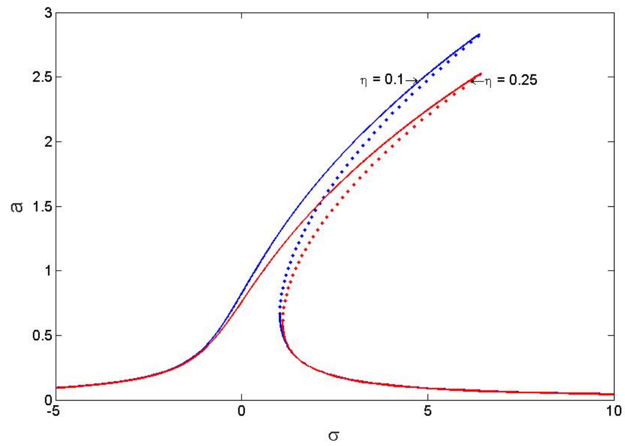

FIGURE 6. frequency response curves: $k = \gamma = 2\pi^4$, $\alpha = 0.5$, $\xi_{s1} = 0.1$ first mode of vibration.

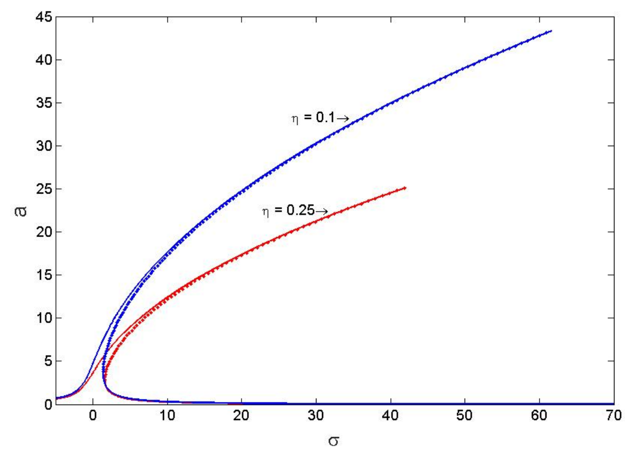

FIGURE 7. frequency response curves: $k = \gamma = 2\pi^4$, $\alpha = 5$, $\xi_{s1} = 0.1$ first mode of vibration.

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