Nonlinear vibration of an axially loaded beam carrying rigid bodies

Cite as: AIP Advances **6**, 125116 (2016); https://doi.org/10.1063/1.4973334
Submitted: 20 May 2016 . Accepted: 13 December 2016 . Published Online: 27 December 2016

O. Barry

AIP ADVANCES **6**, 125116 (2016)

Nonlinear vibration of an axially loaded beam carrying rigid bodies

O. Barry[a]
Assistant Professor, School of Engineering and Technology, Central Michigan University, Mt. Pleasant, Michigan 48859, USA

(Received 20 May 2016; accepted 13 December 2016; published online 27 December 2016)

This paper investigates the nonlinear vibration due to mid-plane stretching of an axially loaded simply supported beam carrying multiple rigid masses. Explicit expressions and closed form solutions of both linear and nonlinear analysis of the present vibration problem are presented for the first time. The validity of the analytical model is demonstrated using finite element analysis and via comparison with the result in the literature. Parametric studies are conducted to examine how the nonlinear frequency and frequency response curve are affected by tension, rotational inertia, and number of intermediate rigid bodies. [http://dx.doi.org/10.1063/1.4973334]

Beam structures are widely used in many engineering applications; such as airplane wings, buildings, bridges, micromechanical systems, overhead transmission lines, as well as many others in the aerospace, mechanical, and civil industries. Numerous authors have studied the transverse vibrations of beams carrying masses or spring-mass-damper system. Most of the studies reported in the literature are based on linear vibration models.[1–9] A few studies can be found on lateral vibration of beams under axial loads.[10–12] It is important to note that the lateral vibrations of beams under tensile axial load are also of practical interest in many engineering applications. In the design of large flexible solar arrays, the boom that supports the array is under pre-tensile stresses due to the tension that must be maintained in the solar cell substrate. Bokaian et al. examined a free vibration analysis for an axially loaded beam with different combinations of boundary conditions.[10] The authors demonstrated that the beam behaves like a string if the dimensionless tension parameter was greater than.[12] Barry et al. studied both free and forced vibration of an axially loaded beam carrying multiple spring-mass-damper system.[11] They presented a generalized orthogonality conditions and showed that using the classical orthogonality condition for the vibration analysis of a loaded beam can lead to erroneous results. All the reported studies so far are based on linear vibration models, which are usually sufficient for predicting the dynamic characteristics of the system when dealing with small deformations. However, when dealing with higher deformation, nonlinearity should be included for accurate modeling. For beam problems under immovable boundary conditions, the most common nonlinearity is attributed to mid-plane stretching. A thorough review of the subject was examined by Nayfeh et al.[13,14] Several authors have also investigated nonlinear vibrations due to mid-plane stretching.[15–22] Burgreen studied the free vibration analysis of pin-ended column.[15] Ozkaya et al. studied the nonlinear vibration of beam with clamped-clamped boundary conditions and carrying one intermediate point mass.[18] They extended their work by investigating the same problem but with various boundary condition[18] and with multiple intermediate point masses.[20] All their works demonstrated a hardening type nonlinearity. The nonlinear vibration of a beam carrying one intermediate spring-mass system was examined by Pakdemirli et al.[21] They postulated that the mid-plane stretching and the spring-mass system had a great effect on the frequency-response curves. Barry et al. extended the work of Pakdemirli et al. by including multiple intermediate mass-spring-damper

[a]barry1o@cmich.edu

support, and various boundary conditions.[22] However, they treated the intermediate masses as point masses therefore neglecting the mass rotational inertia. At this point it is worth mentioning that in all the aforementioned nonlinear vibration references, the authors treated the mass as particles instead of rigid bodies. In the present work, we analytically examined for the first time the nonlinear vibration of an axially loaded beam carrying multiple rigid masses. This work is an extension of our previous work[22] and the work of Ozkaya.[20] We presented explicit expressions for the frequency equation, mode shapes, nonlinear frequency, and the modulation equations for the phase and amplitude. The validity of these analytical expressions were demonstrated through finite element analysis and via comparison with results in the literature. We conducted parametric studies to predict the effect of the mass moment of inertia and tension on the nonlinear frequency and response of the system.

A schematic of the system is depicted in Fig. 1. Following our previous work,[22] the system governing equations are

$$m\ddot{w}_{\mathrm{i+1}} + EIw''''_{\mathrm{i+1}} - Tw''_{\mathrm{i+1}} = \tfrac{EA-T}{2L}\left[\sum_{r=0}^{n}\int_{x_{\mathrm{r}}}^{x_{\mathrm{r+1}}} w'^2_{\mathrm{r+1}}dx\right]w''_{\mathrm{i+1}} \tag{1}$$

for $i = 0, 1, 2, \ldots n$

$$\begin{aligned}
w_{\mathrm{p}}(x_{\mathrm{p}},t) &= w_{\mathrm{p+1}}(x_{\mathrm{p}},t),\\
w'_{\mathrm{p}}(x_{\mathrm{p}},t) &= w'_{\mathrm{p+1}}(x_{\mathrm{p}},t),\\
EI\left(w''_{\mathrm{p}}(x_{\mathrm{p}},t) - w''_{\mathrm{p+1}}(x_{\mathrm{p}},t)\right) + J_{\mathrm{p}}\ddot{w}'_{\mathrm{p}}(x_{\mathrm{p}},t) &= 0\\
EI\left(w'''_{\mathrm{p}}(x_{\mathrm{p}},t) + w'''_{\mathrm{p+1}}(x_{\mathrm{p}},t)\right) - M_{\mathrm{p}}\ddot{w}_{\mathrm{p}}(x_{\mathrm{p}},t) &= 0
\end{aligned} \tag{2}$$

for $\mathrm{p} = 1, 2, 3, \cdots, n$

where w is the transverse displacement of the beam, x is the axial coordinate, m is the mass per unit length of the beam, T is the tension of the beam, EI is the flexural rigidity of the beam, M_{p} and J_{p} are the p^{th} in-span mass and rotational inertia, respectively.

The following dimensionless parameters can be introduced

$$\begin{aligned}
\zeta &= \frac{x}{L}, \quad W_{\mathrm{p}} = \frac{w_{\mathrm{p}}}{L}, \quad \xi_{\mathrm{p}} = \frac{x_{\mathrm{p}}}{L}, \quad \tau = \frac{t}{L^2}\sqrt{\frac{EI}{m}},\\
\eta_{\mathrm{p}} &= \frac{J_{\mathrm{p}}}{mL^3}, \quad \alpha_{\mathrm{mp}} = \frac{M_{\mathrm{p}}}{mL}, \quad R = \sqrt{\frac{I}{AL^2}},\\
s &= \sqrt{\frac{TL^2}{2EI}}, \quad \lambda = \frac{1}{R^2}(1 - 2R^2 s^2),\\
\alpha &= \sqrt{-s^2 + \sqrt{s^4 + \omega^2}}, \quad \beta = \sqrt{s^2 + \sqrt{s^4 + \omega^2}}
\end{aligned} \tag{3}$$

where ω is the circular linear natural frequency. Using the above dimensionless parameters and adding damping and forcing terms, the governing equations becomes

$$\ddot{W}_{\mathrm{i+1}} + W''''_{\mathrm{i+1}} - 2s^2 W''_{\mathrm{i+1}} = \frac{1}{2}\lambda\left[W''_{\mathrm{i+1}}\sum_{r=0}^{n}\int_{\xi_{\mathrm{r}}}^{\xi_{\mathrm{r+1}}} W'^2_{\mathrm{r+1}}d\zeta\right] - 2\bar{\mu}\dot{W}_{\mathrm{i+1}} + \bar{F}_{\mathrm{i+1}}\cos\Omega\tau \tag{4}$$

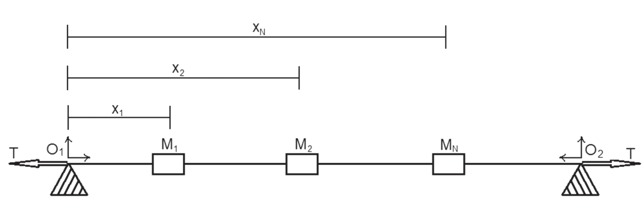

FIG. 1. Schematic of an axially loaded simply supported beam carrying multiple rigid masses.

$$\begin{aligned} W_{\mathrm{p}}(\xi_{\mathrm{p}},\tau) &= W_{\mathrm{p+1}}(\xi_{\mathrm{p}},\tau),\\ W'_{\mathrm{p}}(\xi_{\mathrm{p}},\tau) &= W'_{\mathrm{p+1}}(\xi_{\mathrm{p}},\tau),\\ W''_{\mathrm{p}}(\xi_{\mathrm{p}},\tau) - W''_{\mathrm{p+1}}(\xi_{\mathrm{p}},\tau) &= -\eta_{\mathrm{p}}\ddot{W}'_{\mathrm{p}}(\xi_{\mathrm{p}},\tau),\\ W'''_{\mathrm{p}}(\xi_{\mathrm{p}},\tau) + W'''_{\mathrm{p+1}}(\xi_{\mathrm{p}},\tau) &= \alpha_{\mathrm{mp}}\ddot{W}_{\mathrm{p}}(\xi_{\mathrm{p}},\tau) \end{aligned} \tag{5}$$

where the dots and primes denote differentiation with respect to dimensionless time τ and dimensionless coordinate ξ_{p}, respectively. $\bar{\mu}$ is the dimensionless damping coefficient of the beam, $\bar{F}_{\mathrm{i+1}}$ is the dimensionless excitation amplitude and Ω is the dimensionless excitation frequency.

Due to the absence of quadratic nonlinearity, the solution of Eq. 4 is assumed to be expandable in the form

$$W_{\mathrm{i+1}}(\zeta,\tau,\epsilon) = \epsilon W_{(\mathrm{i+1})1}(\zeta,T_0,T_2) + \epsilon^3 W_{(\mathrm{i+1})3}(\zeta,T_0,T_2) + \ldots, \tag{6}$$

where ϵ is a small dimensionless parameter used for book-keeping. $T_0 = \tau$ is a fast-time scale and $T_2 = \epsilon^2\tau$ is a slow-time scale. The present study considers primary resonances only. Hence, the damping and forcing terms are ordered to counter the effect of the nonlinear terms. The damping coefficient $\bar{\mu}$ and excitation amplitude $\bar{F}_{\mathrm{i+1}}$ are given as

$$\bar{\mu} = \epsilon^2\mu, \quad \bar{F}_{\mathrm{i+1}} = F_{\mathrm{i+1}}\epsilon^3$$

At order ϵ, the problem is linear. Hence, the solution can be assumed as

$$W_{(\mathrm{p})1} = \left[A(T_2)e^{j\omega T_0} + cc\right] Y_{\mathrm{p}}(\zeta) \tag{7}$$

where cc denotes the complex conjugate of the preceding terms and $Y_{\mathrm{p}}(\zeta)$ is the mode shape.

Note that for one intermediate mass, it is more convenient to use two reference frames (i.e., one at each end of the beam) to obtain a more compact representation of the frequency equation and mode shapes. After some algebraic manipulation, the frequency equation for one intermediate mass is obtained as

$$\begin{aligned} &(\alpha\beta^5 + 2\alpha^3\beta^3 + \alpha^5\beta)\sin\alpha\sinh\beta + \alpha_{\mathrm{mp}}\omega^2\left[(\alpha^3 + \alpha\beta^2)\sin\alpha\sinh\beta\xi_1\ \sinh\beta\xi_2\right.\\ &\left.-(\alpha^2\beta + \beta^3)\ \sinh\beta\sin\alpha\xi_1\sin\alpha\xi_2\right] + \alpha_{\mathrm{mp}}\omega^4\eta_{\mathrm{p}}\left[\alpha^2\cos\alpha\xi_1\cos\alpha\xi_2\ \sinh\beta\xi_1\sin\beta\xi_2\right.\\ &+\beta^2\cos\alpha\xi_1\cos\alpha\xi_2\ \cosh\alpha\beta\xi_1\ \cosh\beta\xi_2 + \alpha\beta\,(\sin\alpha\xi_1\cos\alpha\xi_2\ \cosh\beta\xi_1\ \sinh\beta\xi_2\\ &\left.+\ \cos\alpha\xi_1\sin\alpha\xi_2\ \sinh\beta\xi_1\ \cosh\beta\xi_2)\right] + \omega^2\eta_{\mathrm{p}}\left[(\alpha\beta^4 + \alpha^3\beta^2)\sin\alpha\cosh\beta\xi_1\ \cosh\beta\xi_2\right.\\ &\left.-(\alpha^4\beta + \alpha^2\beta^3)\sinh\beta\cosh\alpha\xi_1\cos\alpha\xi_2\right] = 0 \end{aligned} \tag{8}$$

and the mode shapes are

$$Y_{\mathrm{i}}(\zeta) = c_{1\mathrm{i}}\sin\alpha\xi_{\mathrm{i}} + c_{2\mathrm{i}}\sinh\alpha\xi_{\mathrm{i}} \quad \text{for } \mathrm{i} = 1,2 \tag{9}$$

where constants c_{ii} are

$$\begin{aligned} c_{11} &= 1\\ c_{21} &= -\frac{\alpha}{\beta\gamma_{\mathrm{pp}}}\left[(\alpha^2+\beta^2)\sin\alpha\sinh\beta\xi_2\ +\ \omega^2\eta_{\mathrm{p}}\cos\alpha\xi_1(\alpha\cos\alpha\xi_2\sinh\beta\xi_2 - \beta\sin\alpha\xi_2\ \cosh\beta\xi_2)\right]\\ c_{12} &= \frac{1}{\gamma_{\mathrm{pp}}}\left[(\alpha^2+\beta^2)\sin\alpha\xi_1\ \sinh\beta\ +\ \omega^2\eta_{\mathrm{p}}\ \cosh\beta\xi_2(\alpha\cos\alpha\xi_1\ \sinh\beta\xi_1 - \beta\sin\alpha\xi_1\ \cosh\beta\xi_1)\right]\\ c_{22} &= \frac{\alpha}{\beta\gamma_{\mathrm{pp}}}\left[(\alpha^2+\beta^2)\sin\alpha\sinh\beta\xi_1\ +\ \omega^2\eta_{\mathrm{p}}\cos\alpha\xi_2(\alpha\cos\alpha\xi_1\ \sinh\beta\xi_1\beta\sin\alpha\xi_1\ \cosh\beta\xi_1)\right] \end{aligned}$$

and

$$\gamma_{\mathrm{pp}} = [(\alpha^2+\beta^2)\sinh\beta\sin\alpha\xi_2 + \omega^2\eta_{\mathrm{p}}\ \cosh\beta\xi_1(\alpha\cos\alpha\xi_2\ \sinh\beta\xi_2 - \beta\sin\alpha\xi_2\ \cosh\beta\xi_2)]$$

At order ϵ^3, the problem is nonlinear. A solution can be obtained if a solvability condition is satisfied. This condition can be obtained by expressing the solution in the form

$$W_{(\mathrm{i+1})3} = \Phi_{\mathrm{i+1}}(\zeta,T_2)e^{j\omega T_0} + cc + W_{\mathrm{i+1}}(\zeta,T_0,T_2) \tag{10}$$

Following the procedure in our previous work,[22] by substituting Eq. 10 into Eq. 4, multiplying each resulting equation by its corresponding linear mode shape Y_i, taking the integral and adding the two resulting equations, and using the orthogonality condition along with the boundary conditions (after substituting Eq. 10 into Eq. 5), the solvability condition for the nonlinear problem can be obtained as

$$2j\omega(A' + \mu A)b_1 + \frac{3}{2}\lambda A^2\bar{A}b_2^2 - \frac{1}{2}fe^{j\sigma(T_2)} + 2j\omega A'\sum_{r=1}^{n}\alpha_{\mathrm{mr}}Y_{\mathrm{r}}^2(\xi_{\mathrm{r}}) + 2j\omega A'\sum_{r=1}^{n}\eta_{\mathrm{r}}Y_{\mathrm{r}}'^2(\xi_{\mathrm{r}}) = 0 \tag{11}$$

The polar form of the complex amplitude A can be expressed as

$$A = \frac{1}{2}a(T_2)e^{j\theta(T_2)} \tag{12}$$

where a is the real amplitude and θ denotes the phase. Substituting Eq. 12 into Eq. 11 and separating real and imaginary parts yield the following modulation equations for the amplitude and phase

$$\omega ab_4\gamma' = \omega ab_4\sigma - \frac{3}{8}a^3b_3 + \frac{1}{2}f\cos\gamma \tag{13}$$

$$\omega a'b_4 = \frac{1}{2}f\sin\gamma - \omega b_1\mu a \tag{14}$$

where

$$b_1 = \sum_{r=0}^{n}\int_{\xi_{\mathrm{r}}}^{\xi_{\mathrm{r+1}}} Y_{\mathrm{r+1}}^2 d\zeta, \quad b_2 = \sum_{r=0}^{n}\int_{\xi_{\mathrm{r}}}^{\xi_{\mathrm{r+1}}} Y_{\mathrm{r+1}}'^2 d\zeta$$

$$b_3 = \frac{1}{2}\lambda b_2^2, \quad b_4 = b_1 + \sum_{r=1}^{n}\left[\alpha_{\mathrm{r}}Y_{\mathrm{r}}^2(\xi_r) + \eta_{\mathrm{r}}Y_{\mathrm{r}}'^2(\xi_r)\right]$$

$$\gamma = \sigma T_2 - \theta, \quad f = \sum_{r=0}^{n}\int_{\xi_{\mathrm{r}}}^{\xi_{\mathrm{r+1}}} F_{\mathrm{r+1}}Y_{\mathrm{r+1}}d\zeta$$

where σ is a detuning parameter of order.[1] The nonlinear undamped frequencies are obtained from Eqs. 13 and 14 by taking $\mu = f = b_5 = \sigma = 0$

$$\omega_{\mathrm{NL}} = \omega + ka^2 \quad \text{where } k = \frac{3}{8}\frac{b_3}{\omega b_4} \tag{15}$$

In the case of periodic excitation a' and γ' are equal to zero. Hence, the detuning parameter can be expressed as

$$\sigma = \frac{3}{8}\frac{a^2b_3}{\omega b_4} \pm \sqrt{\frac{f^2}{4a^2\omega^2b_4^2} - \frac{b_1^2}{b_4^2}\mu^2} \tag{16}$$

The validity of the frequency equation is demonstrated in Tables I and II. The results in Table I indicate excellent agreement between the present work and the previous work in the literature. Table II shows a comparison between present work and the finite element analysis. The results also show very good agreement with a maximum percentage of error of 0.8%. Table III shows the effect

TABLE I. Present vs. Ref. 19,22; ($\eta = 0, s = 0, \alpha = 1, \xi_1 = 0.5$).

Mode	Present	Ref. 22	Ref. 20
1	5.6795	5.6795	5.6795
2	39.4784	39.4784	39.4784
3	67.8883	67.8883	67.8883
4	157.9144	157.9144	157.9144
4	206.7901	206.7901	206.7901

TABLE II. Analytical vs. FEA (frequency in rad/s); ($\eta = 0.5$, $s = 1$, $\alpha = 0.5$, $\xi_1 = 0.5$).

Mode	Analytical	FEA	% of Error
1	4.9162	4.9559	0.8075
2	7.3091	7.3367	0.3780
3	62.3020	62.6700	0.5907
4	72.2230	72.6290	0.5621
5	200.3700	200.8300	0.2296

TABLE III. First five modes for a beam carrying up to four rotational masses ($s = 1$).

α, η, ξ	Values	Mode	Frequency (rad/s)
$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	1,1,1,1	1	4.2361
[-0.5ex] $\eta_1, \eta_2, \eta_3, \eta_4$	0.5,0.5,0.5,0.5	2	5.6644
[-0.5ex] $\xi_1, \xi_2, \xi_3, \xi_4$	0.1,0.5,0.7,0.9	3	6.0103
		4	9.1118
		5	30.3810

of attaching multiple rigid bodies on the natural frequencies. As expected, the results indicate that the systems natural frequencies decreases with increasing number of intermediate rigid masses.

As for the nonlinear analysis, the validity is demonstrated via comparison of the results in the literature and it is depicted in Fig. 2. The results show an excellent agreement. For validation purpose, the tension is taken to be $s = 0$. As observed in Fig. 2, the curves bend to the right, which is an indication of hardening type nonlinearity. The effect of the tension on the nonlinear natural frequency is depicted in Fig. 3. The results indicate that the stretching of the curve shifts from right to left for $s > 1$. It is also observed that the stretching to the left is more pronounced with increasing tension. Fig. 4 examines the role of the mass rotational inertia on the nonlinear frequency. The results show that the stretching decreases with increasing rotational inertia. In the forced response analysis, the forcing amplitude is $f = 5b_4$ and the damping coefficient is $\mu = 0.2$. The influence of the tension on the frequency response curve is depicted in Fig. 5. As seen previously, the curve tends to bend more to the left with increasing tension. In Fig. 6, the effect of varying the rotational inertia on the frequency response curve is examined. The results show that the frequency response curve tends to bend more

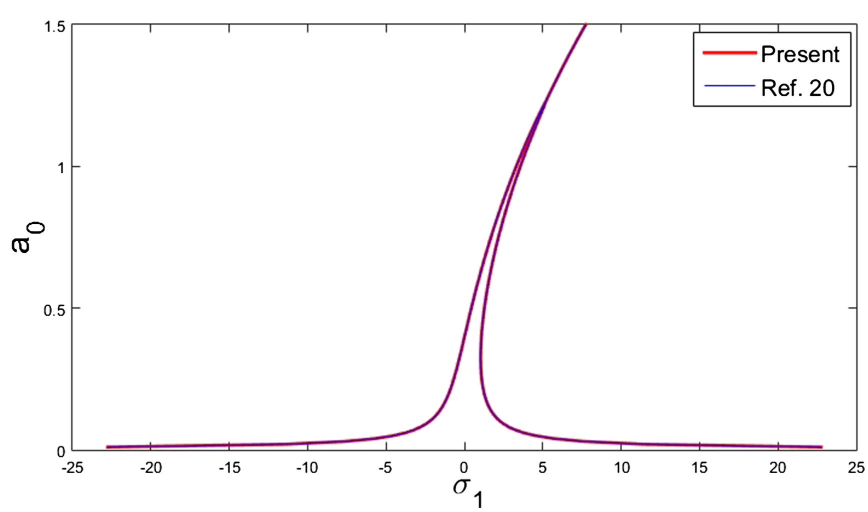

FIG. 2. Validation of the present formulation.

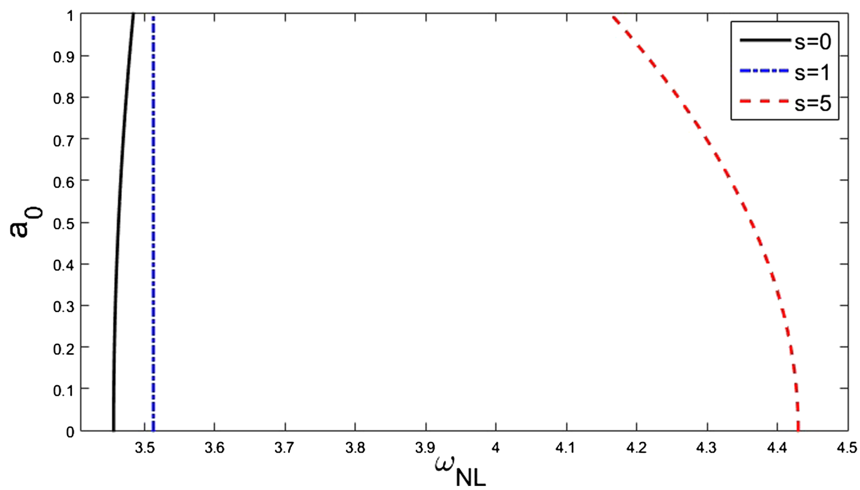

FIG. 3. Effect of tension on the nonlinear frequency for $\xi = 0.5, \alpha_1 = 1, \eta_1 = 0.5$.

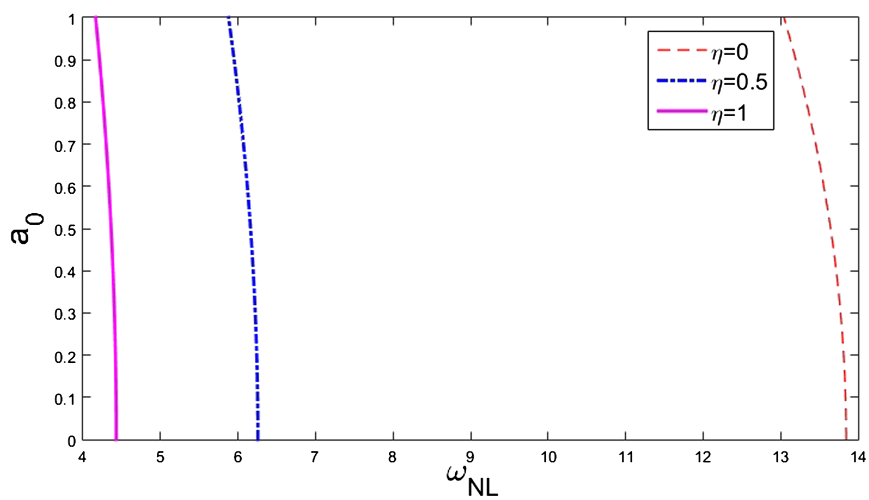

FIG. 4. Effect of rotational inertia on the nonlinear frequency for $\xi = 0.5, \alpha_1 = 1, s = 5$.

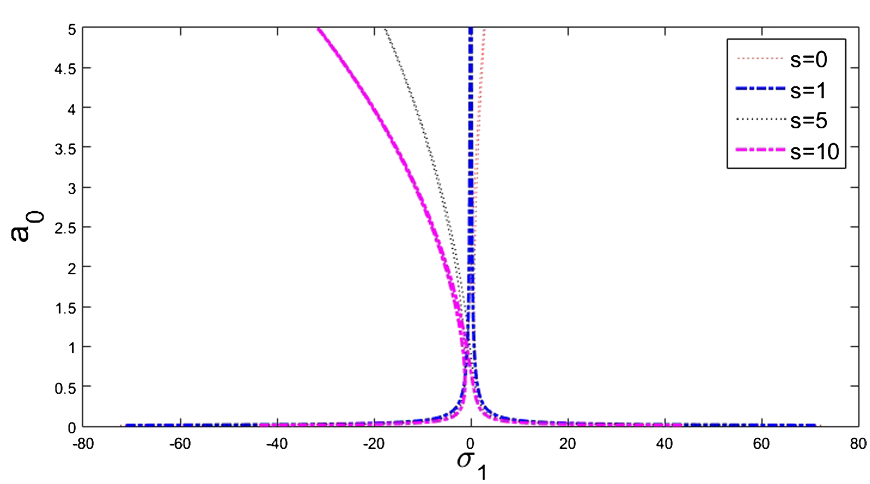

FIG. 5. Effect of tension on the frequency-response-curve for $\xi = 0.5, \alpha_1 = 1, \eta_1 = 0.5$.

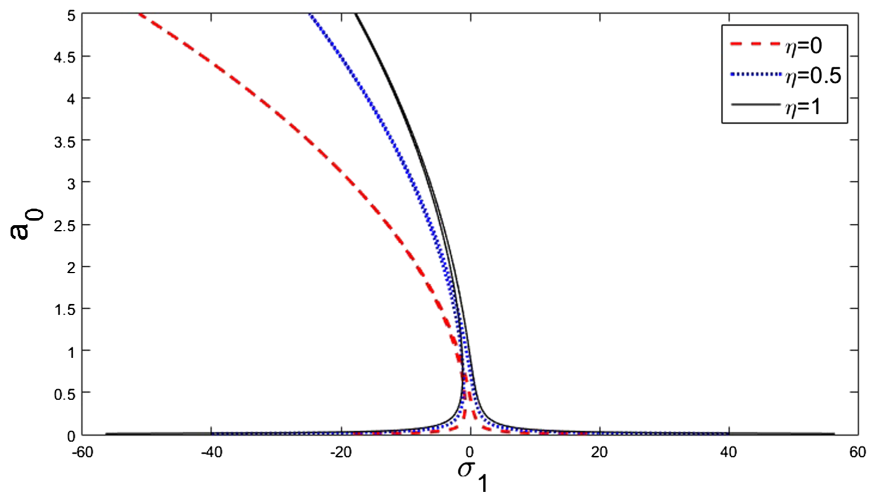

FIG. 6. Effect of rotational inertia on the frequency-response-curve for $\xi = 0.5, \alpha_1 = 1, \eta_1 = 0.5$.

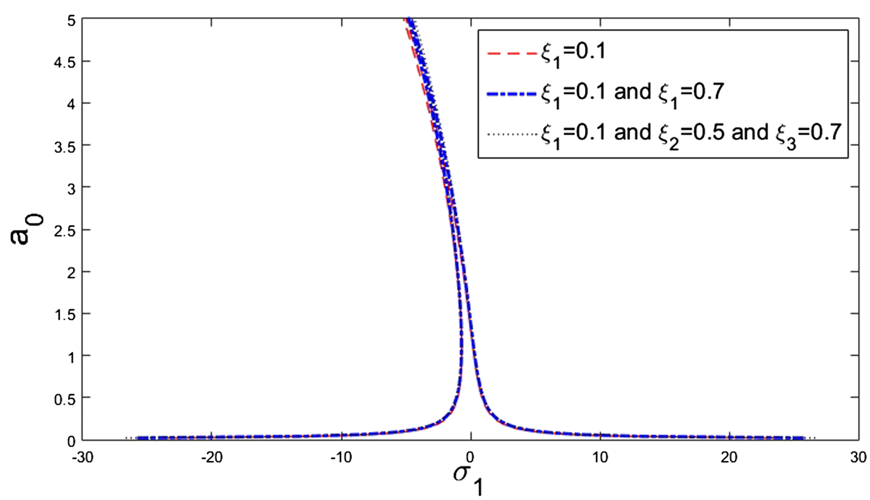

FIG. 7. Effect of multiple rigid masses on the frequency-response-curve for $s = 5$.

to the left with decreasing rotational inertia. This observation is an indication that the tension and the rotational inertia have opposite effect on the frequency response curve. The effect of attaching multiple intermediate rigid bodies is depicted in Fig. 7. The results indicate that the stretching of the frequency response curve tends to decrease as the number of intermediate rigid bodies is increased. This is an indication of the reduction in the softening type nonlinearity. These observations are in agreement with the literature for $s = 0$. In that, the hardening nonlinearity type is more pronounced as the number of intermediate point masses increases.

In conclusion, this paper presents the nonlinear vibration analysis of an axially loaded simply-supported beam carrying multiple intermediate rigid bodies. For the first time, explicit expressions are presented for the characteristic equation, mode shapes, nonlinear frequency, and modulation equations for the steady state phase and steady state amplitude. The validity of the analytical model is demonstrated using finite element analysis and results in the literature. The numerical simulations indicate that the presence of the tension in the beam shifts the nonlinearity type from hardening to softening and that the softening type nonlinearity is more pronounced with increasing tension. However this softening nonlinearity tends to decrease with both increasing mass rotational inertia and increasing number of intermediate rigid bodies.

[1] M. Gürgöze, Journal of Sound and Vibration **217**, 585 (1998).
[2] H.-Y. Lin and Y.-C. Tsai, Journal of Sound and Vibration **302**, 442 (2007).
[3] P. Hassanpour, W. Cleghorn, J. Mills, and E. Esmailzadeh, Journal of Vibration and Control **13**, 1723 (2007).
[4] J.-S. Wu and C.-G. Huang, Communications in Numerical Methods in Engineering **11**, 743 (1995).
[5] S. Naguleswaran, International Journal of Mechanical Sciences **43**, 2737 (2001).
[6] C. Kirk and S. Wiedemann, Journal of Sound and Vibration **254**, 939 (2002).
[7] S. O. Maiz, D. Bambill, C. Rossit, and P. Laura, Journal of Sound and Vibration **303**, 895 (2007).
[8] C. Chang, Journal of Sound and Vibration **237**, 733 (2000).
[9] J.-S. Wu and T.-L. Lin, Journal of Sound and Vibration **136**, 201 (1990).
[10] A. Bokaian, Journal of Sound and Vibration **142**, 481 (1990).
[11] O. Barry, D. Oguamanam, and J. Zu, Shock and Vibration **2014** (2014).
[12] S. Woinowsky-Krieger, J. Appl. Mech **17**, 35 (1950).
[13] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations* (John Wiley & Sons, 2008).
[14] A. H. Nayfeh, *Introduction to Perturbation Techniques* (John Wiley & Sons, 2011).
[15] D. Burgreen, "Free vibrations of a pin-ended column with constant distance between pin ends." *Tech. Rep.* (DTIC Document, 1950).
[16] R. Mestrom, R. Fey, K. Phan, and H. Nijmeijer, Sensors and Actuators A: Physical **162**, 225 (2010).
[17] K. Low, Journal of Sound and Vibration **207**, 284 (1997).
[18] E. Özkaya, M. Pakdemirli, and H. Öz, Journal of Sound and Vibration **199**, 679 (1997).
[19] E. Özkaya and M. Pakdemirli, Journal of Sound and Vibration **221**, 491 (1999).
[20] E. Özkaya, Journal of Sound and Vibration **257**, 413 (2002).
[21] M. Pakdemirli and A. Nayfeh, Journal of Vibration and Acoustics **116**, 433 (1994).
[22] O. Barry, D. Oguamanam, and J. Zu, Nonlinear Dynamics **77**, 1597 (2014).