Proceedings of the ASME 2019 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2019
August 18-21, 2019, Anaheim, USA

IDETC2019-98414

ON THE SPECTRO-SPATIAL WAVE FEATURES IN NONLINEAR METAMATERIALS WITH MULTIPLE LOCAL RESONATORS

Mohammad A. Bukhari
Department of Mechanical Engineering
Virginia Tech, Blacksburg, Virginia 24060
Email: bukhari@vt.edu

Oumar R. Barry*
Department of Mechanical Engineering
Virginia Tech, Blacksburg, Virginia 24060
Email: obarry@vt.edu

ABSTRACT

Considerable attention has been given to nonlinear metamaterials because they offer some interesting phenomena such as solitons, frequency shifts, and tunable bandgaps. However, only little is known about the spectro-spatial properties of a wave propagating in nonlinear periodic chains, particularly, a cell with multiple nonlinear resonators. This problem is investigated here. Our study examines both hardening and softening nonlinearities in the chains and in the local resonators. Explicit expressions for the nonlinear dispersion relations are derived by the method of multiple scales. We validate our analytical results using numerical simulations. The numerical simulation is based on spectro-spatial analysis using signal processing techniques such as spatial-spectrogram and wave filtering. The spectro-spatial analysis provides detailed information about the interactions of dispersive and nonlinear phenomena of waveform in both short and long-wavelength domains. The findings suggest that nonlinear resonators can have more effect on the waveform than the nonlinear chains. This observation is valid in both short and long wavelength limits.

INTRODUCTION

Introducing unique dynamic properties artificially from engineering configurations and material constituent leads to promising materials with exceptional characteristics in different engineering applications. These materials, which are called metamaterials, have attracted many researchers because of their wider applications in different fields. They were first introduced in electromagnetic and optical wave propagation and later extended to mechanical waves applications [1, 2].

Mechanical metamaterials are often fabricated from periodic cells arranged carefully. The earlier study for periodic structure was in the 1900s [3–8]. These structures form a bandgaps due to Bragg scattering at wavelengths near their lattice constant. This enables, for example, vibration attenuation at low frequencies located inside the bandgap. However, the condition associated with Brag scattering makes this application limited to large structures.

Attaching local resonators on the crystal allows a bandgap formation at wavelengths much larger than the lattice constant [9]. This enables the vibration control of small structures at low frequencies, thus widening the possible applications of metamaterials. Further investigation on the comparison between local resonator and Bragg scattering concepts can be found in [10]. Multiple bandgaps at different frequency ranges can also be developed by using multiple resonators with different parameters [11, 12].

Beyond vibration suppression, nonlinear metamaterials offer a wide pool of applications including gap solitons [13], dark solitons, envelope and dark solitons [14], wave non-reciprocity [15], and altering band structure limits [16].

Weakly nonlinear acoustics metamaterials were investigated analytically by using different perturbation techniques [17, 18]. For instance, Narisetti et al. [19], employed the Lindstedt-Poincare method in deriving the dispersion relations for nonlinear chain and validated the results numerically. The method of multiple scales can deal with more complicated nonlinear sys-

*Address all correspondence to this author.

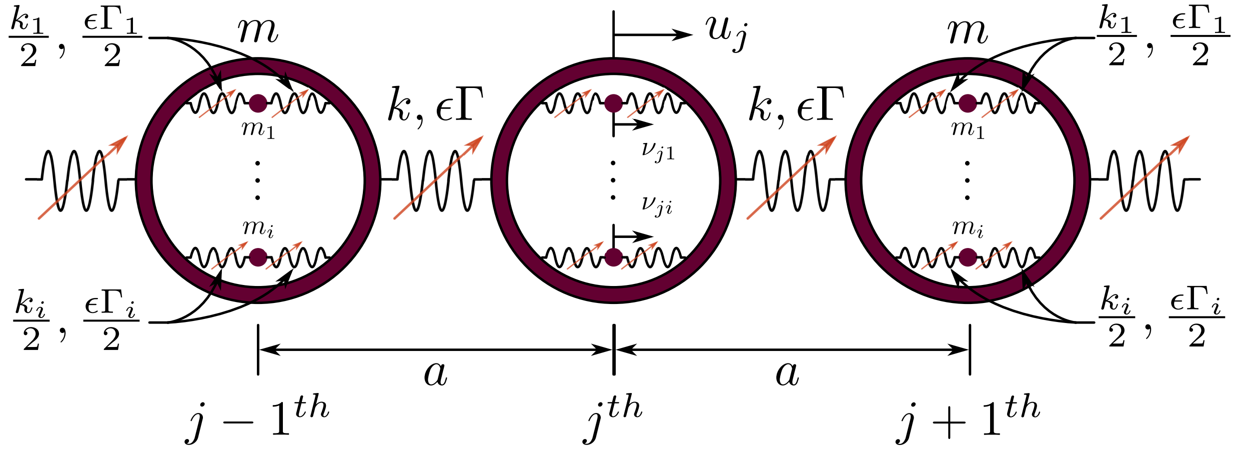

FIGURE 1: A schematic diagram for the nonlinear acoustics metamaterial

tems like multiple waves interaction [20].

Wave non-reciprocity can be used in uni-direction acoustics wave propagation (e.g. Acoustics diode). This can be obtained by coupling linear and nonlinear mediums [15, 21, 22], bifurcation in granular structures [23], or nonlinear hierarchical internal structures [24]. Moreover, acoustics rectifier can be obtained by a cubically weakly nonlinear oscillator attached to a linear periodic lattice such that the operation frequencies of the rectifier coincide with the secondary resonances of the nonlinear oscillator [25].

Analyzing nonlinear metamaterials is often performed by tracking the change in the temporal state properties and discussing the existence of solitary waves, and dispersion characteristics. However, Ganesh and Gonella [26] have studied the spectro-spatial wave packet propagation features of nonlinear periodic chains by some signal processing tools. This allows detecting wave localization (born of solitons), and reconstructing dispersion curves. However, although their analytical expressions could predict the shift in dispersion curves, many other nonlinear phenomena could not be inferred. Zhou et al. [27], extended Ganesh and Gonella's work by including local linear resonators and studied the spectro-spatial wave features of nonlinear acoustic metamaterial. In both studies [26, 27], the effect of nonlinearity in the chain was limited to short wavelength region only. None of the studies included the nonlinearity in local resonators and determine how nonlinear resonators affect dispersion characteristics or propagation of solitary waves in both long and short-wavelength domains.

Seeking for a nonlinear system that offers interesting wave propagation phenomena in all wavelength regions, which is a rare find, we investigate for the first time the nonlinear vibration of a nonlinear chain with multiple nonlinear local resonators analytically and numerically. The nonlinearity is assumed to be weakly cubic type with softening or hardening nonlinear coefficients. In one case, we study the effect of nonlinearity attributed to the nonlinearity in the chain only. In another case, we examine the nonlinearity effect caused by the local resonator only. We employ the method of multiple scales to generate approximate close form expressions for the dispersion curves of a nonlinear (linear) chain with any number of linear (nonlinear) resonators. We follow this by numerical simulations of the metastructure subjected to a wave packet input impulse. The results are used to check our analytical model in predicting the cut-off frequency. We then use multiple signal processing tools in order to investigate the spectro-spatial properties of the nonlinear acoustic metamaterial. Finally, we study the effect of both hardening and softening nonlinearities in the chain and in the local resonators. The findings suggest that very interesting dispersion characteristics and propagation of solitary wave can be realized in both long-wavelength and short-wavelength domains using nonlinear chain with multiple nonlinear local resonators. These interesting wave propagation characteristics can be employed to design superior vibration isolation and acoustic diode devices.

SYSTEM DESCRIPTION AND MATHEMATICAL MODELING

A schematic diagram for the acoustic metamaterial chain is depicted in Fig. 1. The chain consists of periodic cells. Each cell is represented by a mass, m, and it is connected to the other cells by a linear or nonlinear spring with linear coefficient, k, and nonlinear coefficient $\varepsilon\Gamma$. There are s number of local resonators in each cell. The i^{th} resonator consists of a mass, m_i and connected to the j^{th} cell by a linear or nonlinear spring with linear coeficient, k_i, and a nonlinear coefficient, $\varepsilon\Gamma_i$. It is noteworthy here that the system is reduced to a linear system if $\varepsilon = 0$.

The equations of motion for the n^{th} cell can be expressed as follows

$$m\ddot{u}_n + k(2u_n - u_{n-1} - u_{n+1}) + \varepsilon\Gamma((u_n - u_{n-1})^3 + (u_n - u_{n+1})^3) + \sum_{i=1}^{s} k_i(u_n - v_{ni}) + \sum_{i=1}^{s} \varepsilon\Gamma_i(u_n - v_{ni})^3 = 0 \quad (1)$$

$$m_i\ddot{v}_{ni} + k_i(v_{ni} - u_n) + \varepsilon\Gamma_i(v_{ni} - u_n)^3 = 0 \quad (2)$$

For the case of nonlinear chain only, we set $\Gamma_i = 0$ while we set $\Gamma = 0$ in the case of nonlinear resonator only.

Eqns. (1)-(2) can be written in the non-dimensional form as

$$\ddot{u_n} + 2u_n - u_{n-1} - u_{n+1} + \varepsilon\bar{\Gamma}((u_n - u_{n-1})^3 + (u_n - u_{n+1})^3) + \sum_{i=1}^{s} \bar{k}_i(u_n - v_{ni}) + \sum_{i=1}^{s} \varepsilon\bar{\Gamma}_i(u_n - v_{ni})^3 = 0 \quad (3)$$

$$\frac{\omega_n^2}{\omega_{di}^2}\ddot{v}_{ni} + (v_{ni} - u_n) + \varepsilon\bar{\Gamma}_i(v_{ni} - u_n)^3 = 0 \quad (4)$$

where the dots here denote the derivative in terms of the non-dimensional time $\tau = \omega_n t$, $\bar{\Gamma} = \frac{\Gamma}{\omega_n^2 m}$, $\bar{k}_i = \frac{k_i}{\omega_n^2 m}$, $\omega_n^2 = k/m$, and $\omega_{di}^2 = k_i/m_i$.

Approximate Analytical Solution by the Method of Multiple Scales

For weakly nonlinear systems like the one presented in Eqns. (3)-(4), perturbation techniques can be employed to obtain approximate analytical solution of the dispersion curves. Here we use the method of multiple scales to present explicit expressions for the dispersion relations. The method of multiple scales is advantageous over other techniques due to the simplicity of handling and collecting the secular terms in multiple equations or complicated systems. The approximate solution can be represented up to second order approximation as

$$u_n(t,\varepsilon) = u_{n0}(T_0,T_1) + \varepsilon u_{n1}(T_0,T_1) \quad (5)$$

$$v_{ni}(t,\varepsilon) = v_{ni0}(T_0,T_1) + \varepsilon v_{ni1}(T_0,T_1) \quad (6)$$

where $T_0 = \tau$ is the fast time scale and $T_1 = \varepsilon\tau$ is the slow time scale. The system can now be represented by two independent variables (scales) and applying the full derivative is not valid any more. Instead, we can represent the time derivative by the chain rule as

$$(\ddot{\ }) = D_0^2 + 2\varepsilon D_0 D_1 + ... \quad (7)$$

where $D_n = \frac{\partial}{\partial T_n}$.

The solution of the linear system can be expressed as

$$u_n = Ae^{i(n\bar{k} - \omega T_0)} + c.c \quad (8)$$

$$v_{ni} = B_i e^{i(n\bar{k} - \omega T_0)} + c.c \quad (9)$$

where $\bar{k} = aq$ is the nondimensional wavenumber. For convenience, we drop the bar from $\bar{k}$ in the subsequent analysis since the linear stiffness of the chain k does not appear any more in the normalized parameters.

Substituting Eqns. (5)-(7) into Eqns. (3)-(4), collecting the coefficients of $\varepsilon^0 \& \varepsilon$, and then substituting Eqns. (8)-(9), (refer to [28] for more details) one can write the linear dispersion relation for all cases of nonlinearity as

$$-\omega^2 + (2 - 2\cos k) + \sum_{i=1}^{s} \bar{k}_i(1 - K_{\omega i}) = 0 \quad (10)$$

where $K_{\omega i} = \frac{1}{1-\omega_n^2\omega^2/\omega_{di}^2}$. To derive the nonlinear solution, the vibration amplitude should be written in the polar form as

$$A = \frac{1}{2}\alpha e^{i\beta} \quad (11)$$

and solving for the amplitude α, reveals that $\alpha = \alpha_0$, where α_0 is a constant, for both cases of nonlinearity. The phase can be written for each case as

Nonlinear chain $\bar{\Gamma} \neq 0$

$$\beta = -\frac{3\bar{\Gamma}\alpha^2(1-\cos k)^2}{2\omega\left(1 + \sum_{i=0}^{s} \frac{\bar{k}_i\omega_n^2/\omega_{di}^2}{1-\omega_n^2\omega^2/\omega_{di}^2} K_{\omega i}\right)} T_1 \quad (12)$$

Nonlinear resonator $\bar{\Gamma}_i \neq 0$

$$\beta = -\frac{\sum_{i=1}^{s}\left[\frac{3}{8}\alpha^2(1-K_{\omega_i})^3\bar{\Gamma}_i\left(\frac{\bar{k}_i}{1-\omega_n^2\omega^2/\omega_{di}^2} - 1\right)\right]}{\omega\left(1 + \sum_{i=1}^{s} \frac{\bar{k}_i\omega_n^2/\omega_{di}}{1-\omega_n^2\omega^2/\omega_{di}^2} K_{\omega i}\right)} T_1 \quad (13)$$

Therefore, the nonlinear dispersion curves can be written as

$$\omega_{nl} = \omega + \varepsilon\beta' \quad (14)$$

where β' is the derivative in terms of the slow time scale.

From Eqn. (12), it can be observed that the nonlinear frequency in the nonlinear chain case is a function of wavenumber. In fact, the correction factor $\beta \sim 0$ when k is very small and hence the effect of chain nonlinearity (β) is negligible for long wavelength limit at both acoustic and optical modes. On the other hand, for the case of the nonlinear resonator (Eqn. (13)), the wavenumber does not explicitly appear in the expression of the correction factor and hence the only wavenumber dependence in this case is through the linear dispersion relation (i.e. Eqn. (10)).

Also note the appearance of a new term $(1-K_{\omega i})^3$ in the numerator of Eqn. (13), which can have a significant effect on the correction factor β and hence on the nonlinear frequency when the resonator is tuned to the excitation frequency regardless of the wavenumber. It is noteworthy here that the derived expression for, β in Eqn. (13), is correct and different from that obtained in [16], since the latter omitted the contribution of the resonators on the left hand side from the equations at order ε [19, 20] (for more information refer to [29]).

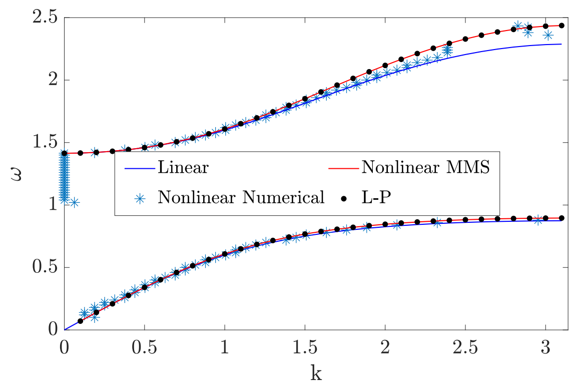

FIGURE 2: Validating the results of nonlinear chain with single linear resonator, $\varepsilon\bar{\Gamma}\alpha^2 = 0.06$, $\varepsilon\bar{\Gamma}_1\alpha^2 = 0$.

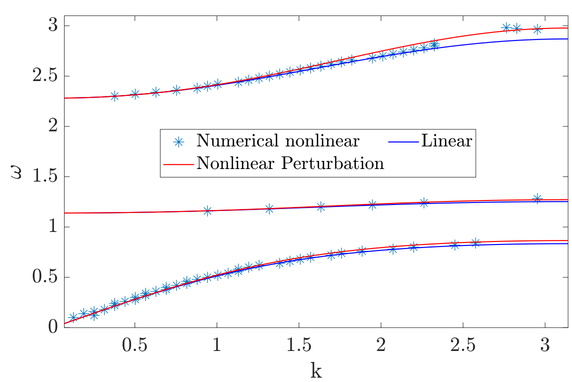

FIGURE 3: Validating the results of nonlinear chain with two linear resonators, $\varepsilon\bar{\Gamma}\alpha^2 = 0.06$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$.

VALIDATING ANALYTICAL RESULTS

To validate the dispersion relations obtained by the method of multiple scale, we compare the current results with those obtained in the literature for a nonlinear chain single linear resonator system obtained by Lindstedt-Poincare methods and with

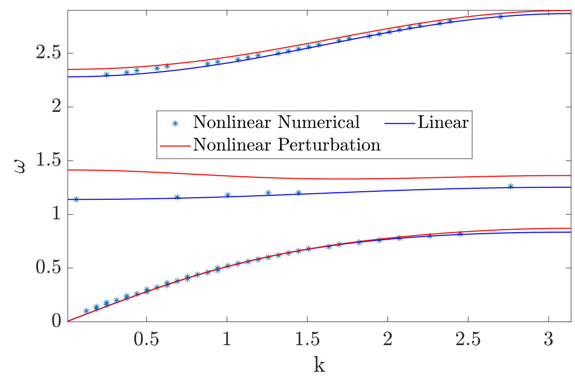

FIGURE 4: Validating the results of linear chain with two nonlinear resonators, $\varepsilon\bar{\Gamma}\alpha^2 = 0$, $\varepsilon\bar{\Gamma}_1\alpha^2 = 0.06$, $\varepsilon\bar{\Gamma}_2\alpha^2 = 0$.

those obtained numerically. For this part, we select $\omega_n = \omega_{d1} = 1000$ rad/sec, $\bar{k}_i = 1$, $s = 1$, $\varepsilon\bar{\Gamma}\alpha^2 = 0.06$, and $\varepsilon\bar{\Gamma}_i\alpha^2 = 0$.

For numerical simulations, we simulate a chain consisting of 500 cells and attached to it s number of resonators (e.g. $s = 1$ in the first part of validation, then we set $s = 2$). The boundaries of the chain are assumed to be a perfectly matched layer (PML) in order to absorb and dissipate incoming waves, as well as, minimize wave reflections at each end [19]. The system is excited by a transient wave packets signal at different wavenumbers. The velocity of the wave packet is selected to limit the motion of the signal in one direction and suppress any waves traveling in the opposite direction [26]. Therefore, the system is excited at first cell while the wave will travel to the other end (cell 500). The numerical integration is done by MATLAB built in integrator ODE45. After running the simulation at a specific wavenumber, 2-D Fast Fourier Transform is applied on the displacement matrix and the frequency and wavenumber corresponding to the maximum amplitude value are collected. These values represent the point in the dispersion curve corresponding to the wavenumber of excitation signal [20]. The wavenumber is then swept to reconstruct other points in the dispersion curves numerically.

Fig. 2 presents a comparison between our results, the literature results, and numerical results. Our multiple scales results show very good agreement for the case of nonlinear chain with single linear resonator.

For the case of nonlinear chain with multiple linear resonators, we validate our analytical results using numerical simulation only since the literature lacks simulations for similar nonlinear systems. The results are shown in Fig. 3 for the case of two resonators where $\omega_{d1} = \omega_n$ and $\omega_{d2} = 1.5\omega_n$. The results show that the method of multiple scale can accurately predict, in general, dispersion curves and the trend of this type of nonlinearity. However, it fails to predict any other nonlinear dynamics phenomena such as solitons and the presence of secondary resonances as we will show in the following sections.

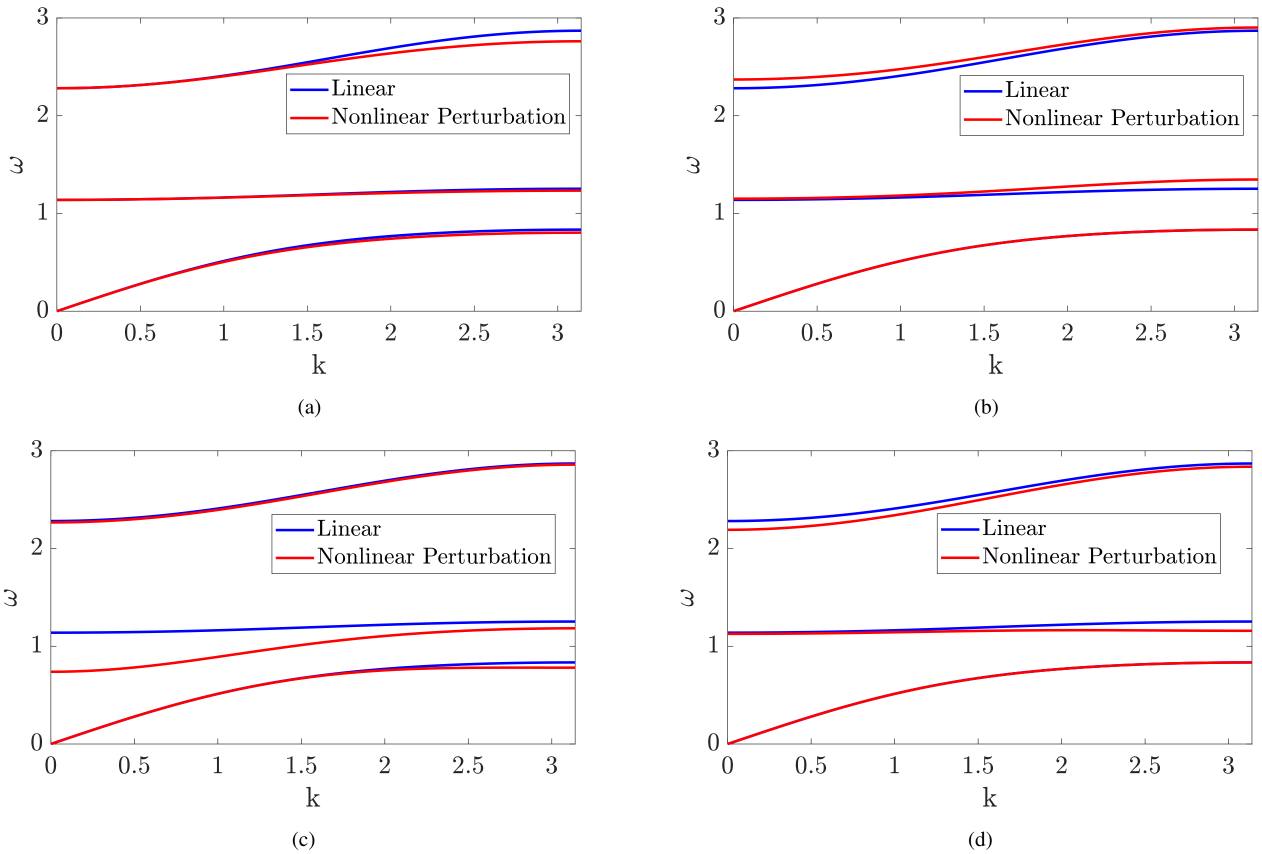

FIGURE 5: Analytical dispersion curves for acoustics metamaterial and two local resonators with different types and sources of nonlinearities: (a) Softening chain nonlinearity $\varepsilon\bar{\Gamma}\alpha^2 = -0.06$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$; (b) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = 0.06$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$; (c) Softening resonator nonlinearity $\varepsilon\bar{\Gamma}_1\alpha^2 = -0.06$, $\varepsilon\bar{\Gamma}\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$; (d) Softening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = -0.06$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$.

Furthermore, the numerical and analytical results of the nonlinear resonator are plotted in Fig. 4. We can observe that the method of multiple scales is a good predictor of the upper and lower branches of the dispersion curve, but a poor predictor of the middle branch when the natural frequency of the system is $\omega_{d1} = \omega_n$. Therefore, this region should be handled by a different approach.

THE EFFECT OF DIFFERENT TYPES OF NONLINEARITIES ON THE BANDGAP BOUNDARIES

After checking the obtained solution for each case, we examine the effect of nonlinearity on the wave propagation in various wavelength regions. In addition to Figs. 3-4, we present the analytical dispersion curves for different kind and source of nonlinearities in Fig. 5.

We can observe from Fig. 3 and Fig. 5.(a) that the nonlinear chain affects mainly the short wavelength region. The effect of nonlinearity in the long wavelength region is almost negligible; however, a significant shift of the dispersion curves is observed at high wavenumbers. On the other hand, Fig. 4 and Figs. 5.(b)-(d) show that systems with nonlinear resonators has significant impact on the dispersion curves in the long wavelength region.

Moreover, it is demonstrated that the effect of nonlinear resonators becomes more pronounced at frequencies near the resonator frequency. For instance, in Fig. 4 and Fig. 5.(c), a significant shift occurs near the resonance frequency of the nonlinear resonators $\omega_{d1} = \omega_n$. However, making the second resonators $\omega_{d2} = 1.5\omega_n$ nonlinear, shifts the effect of nonlinearity to other frequency regions.

It is also demonstrated that tuning the bandgap can be done by changing the type of nonlinearity. In Fig. 5.(a) and Figs. 5.(c)-(d), softening nonlinearity shifts the dispersion curves lower, thus increasing the size of the bandgap. On the other hand, hardening nonlinearity shifts the dispersion curves up as shown in Figs. 3-4 and Fig. 5.(b).

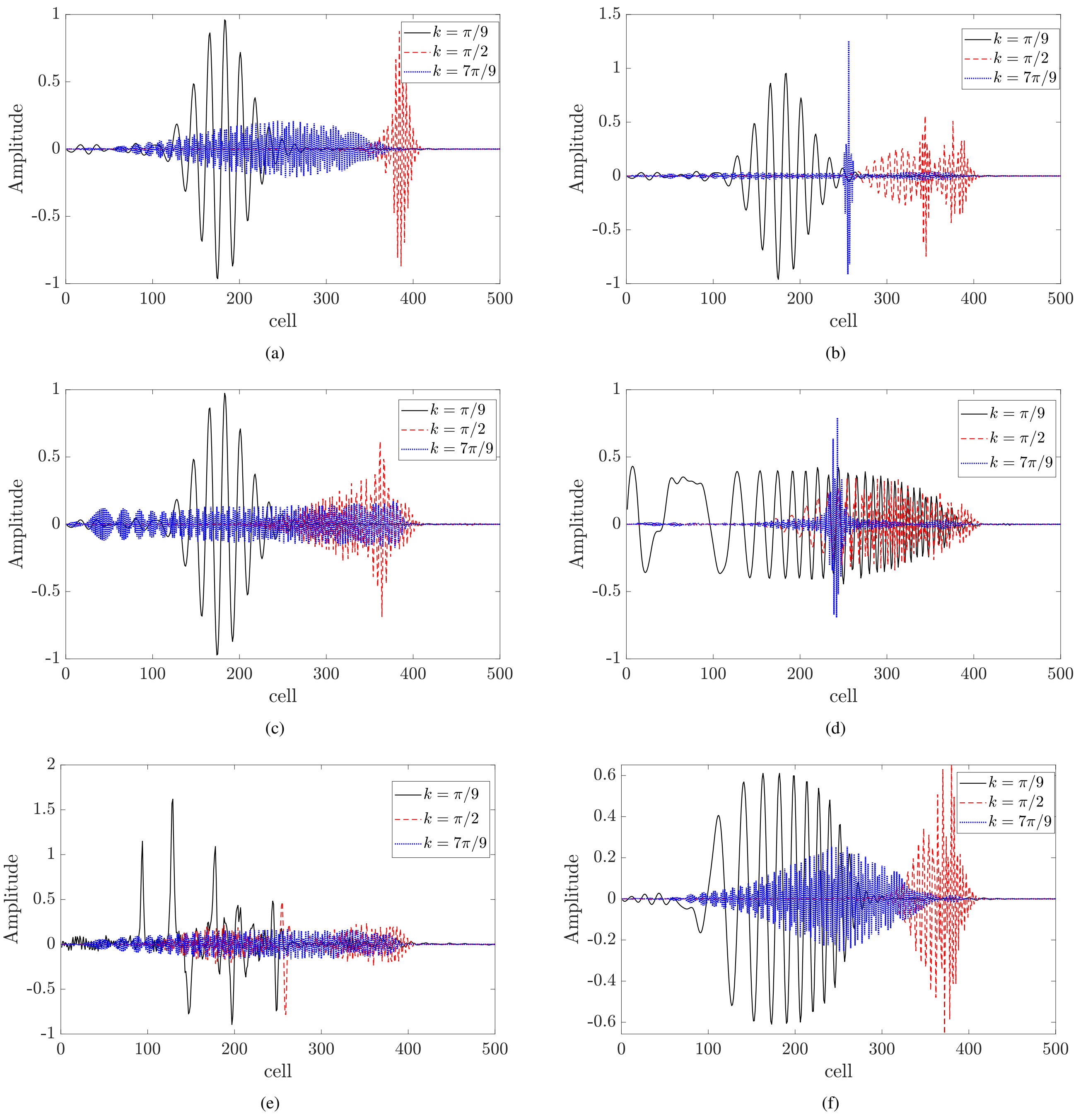

FIGURE 6: Spatial profile of the wave packet for different types and sources of nonlinearities at frequencies in the upper branch of dispersion curve: (a) Linear chain $\varepsilon\bar{\Gamma}\alpha^2 = \varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$; (b) Hardening chain nonlinearity $\varepsilon\bar{\Gamma}\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$; (c) Softening chain nonlinearity $\varepsilon\bar{\Gamma}\alpha^2 = -0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$; (d) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$; (e) Softening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = -0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$; (f) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_1\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_2\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$.

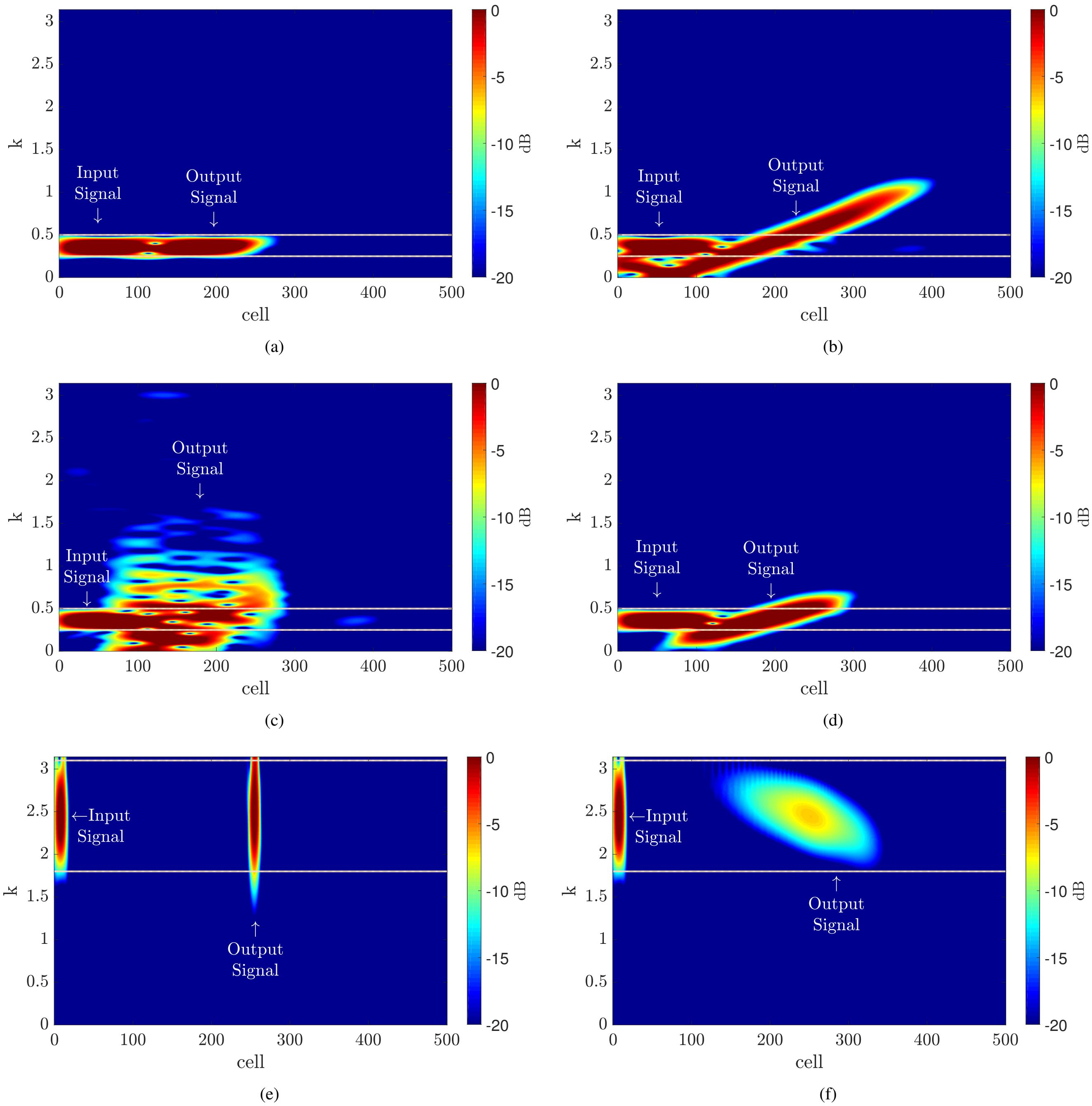

FIGURE 7: Spatial spectrograms of the wave packet for different types and sources of nonlinearities at frequencies in the upper branch of dispersion curve: (a) Hardening chain nonlinearity $\varepsilon\bar{\Gamma}\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$, $k = \pi/9$; (b) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = \pi/9$; (c) Softening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = -0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = \pi/9$; (d) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_1\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_2\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = \pi/9$; (e) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = 7\pi/9$; (f) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_1\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_2\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = 7\pi/9$.

SPECTRO-SPATIAL ANALYSIS

Although the cut-off frequencies can be predicted by the method of multiple scales, other nonlinear wave propagation features cannot be characterized. This suggests the use of spectro-spatial analysis to characterize the wave propagation in the proposed metastructure. It should be noted that all the following simulations are based on the optical branch because this branch is more affected by nonlinearity than the acoustic branch. Also the numerical simulation for the optical mode is much faster. Moreover, Figs. 6-7 are plotted at the end of numerical simulations. The numerical simulations lasted for $t = 8$ sec.

The spatial profile of the wave packet are depicted in Fig. 6 for different types of nonlinearities. We observe that the nonlinear chain gives rise to wave localization with increasing wavenumber when the nonlinearity is hardening Fig. 6.(b) and wave dispersive when the nonlinearity is softening Fig. 6.(c). However, Figs. 6.(a)-(c) indicate that nonlinear hardening chain has no effect on the wave profile in long wavelength region for both types of nonlinearities . The opposite is observed in Figs. 6.(d)-(f) when the system is changed to linear chains with nonlinear local resonators. It is evident that the wave profile is distorted in all wavelength domains. In Fig. 6.(d), a hardening resonator exhibits dispersive wave at long wavelength and localized wave at short wavelength. On the other hand, a softening resonator shows an interesting behavior at long wavelength limit since the wave profile has localized and dispersive components; however, the localized component vanishes reducing wavelength (i.e. increasing wavenumber) as shown in Fig. 6.(e). This effect of resonator nonlinearity depends significantly on the frequency of the nonlinear resonator. For example, tuning the nonlinear resonator away from the upper dispersion curve results in significant reduction in the effect of nonlinear wave propagation phenomena, specifically, in the short wavelength region as shown in Fig. 6.(d). It is noteworthy that, albeit the analytical dispersion curves fail to predict the cut-off frequency and other important wave propagation features, they accurately predict how the nonlinearities in both the chains and resonators affect the wave propagation across all wavelength domains. In that, their predictions about the effect of nonlinearities agree with the spatial profile plots. For example, both Fig. 3 and Fig. 6.(b) show hardening chains to have no effect in long-wavelength domain and significant effect in short wavelength domain.

Fig. 7 shows the spectrograms of the wave propagating through the metastructure in both short and long-wavelengths. As we observed before, the nonlinear chain has no effect on the structure in the long wavelength limit. This is clearly shown in Fig. 7.(a), the output wave profile is exactly the same as the input signal. However, as shown in Figs. 7.(b)-(c), a significant distortion to the input signal is observed when we change the nonlinearity from chain to local resonator. The signal becomes clearly dispersive along the chain with significant equal amplitude when the nonlinearity is hardening as shown in Fig. 7.(b). When the nonlinearity is of softening type, we observe multiple localized signals, as well as, dispersive components (Fig. 7.(c)). The dispersive components are generated at wide range of wavenumbers outside the initial signal wavenumber content. In the short wavelength region, the effect of nonlinear resonator is similar to that of nonlinear chain, the output signal is localized unlike in the linear case where the signal at this limit is completely dispersive. This indicates that soliton formation is also possible in the case of nonlinear resonator as shown in Fig. 7.(e). In Fig. 7.(d) and Fig. 7.(f), it is observed that a nonlinear resonator with frequency away from the excitation frequency has less effect on the wave profile, specifically, in the short wavelength limit where it is completely linear.

Finally, we present the effect of nonlinear resonators in the image of 2-D fast Fourier transform depicted in Fig. 8. The linear signal is similar to the nonlinear signal as shown in Fig. 8.(a), thus confirming that the nonlinear chain has no effect in this limit. In Fig. 8.(b)-(c), the nonlinear resonator shows a wider distribution of the signal along both the frequency and wavenumber ranges in the long wavelength limit for both types of nonlinearity. This observation suggests that such nonlinear acoustic metamaterial can be suitable for applications such as acoustic diode. Fig. 8.(e) demonstrates that the nonlinear resonator is also effective in the short wavelength limit since it localizes the signal and stretches it over a wider region. However, it is also demonstrated in Fig. 8.(d) and Fig. 8.(f) that the effect of nonlinear resonator vanishes when it is not tuned carefully. Overall, both spectral (wavenumber-frequency domain) and topological/physical (space-time domain) analyses provide good insight about the nonlinear effect on wave propagation across all wavelength regions. But only the topological analysis can provide detail information about the physical features of wave propagation such as solitons formation.

CONCLUSION

In this paper, a nonlinear acoustics metamaterial with multiple local resonators was investigated analytically and numerically. In one case, we examined the nonlinearity in the chains and in another we investigated the nonlinearity in the resonators. Closed-form expressions were presented for the nonlinear dispersion relations using the method of multiple scales. These expressions are more general since they can be applied for nonlinear chains with any number of nonlinear local resonators. The analytical results were validated via comparison with those in the literature and those obtained numerically. The validation revealed that the analytical results can predict the cut-off frequency in both cases; however, it fails to predict the dispersion curve near the resonator frequency. The analytical dispersion equation for the case of nonlinear resonator shows a significant shift at all wavelength limits, particularly when the excitation frequency

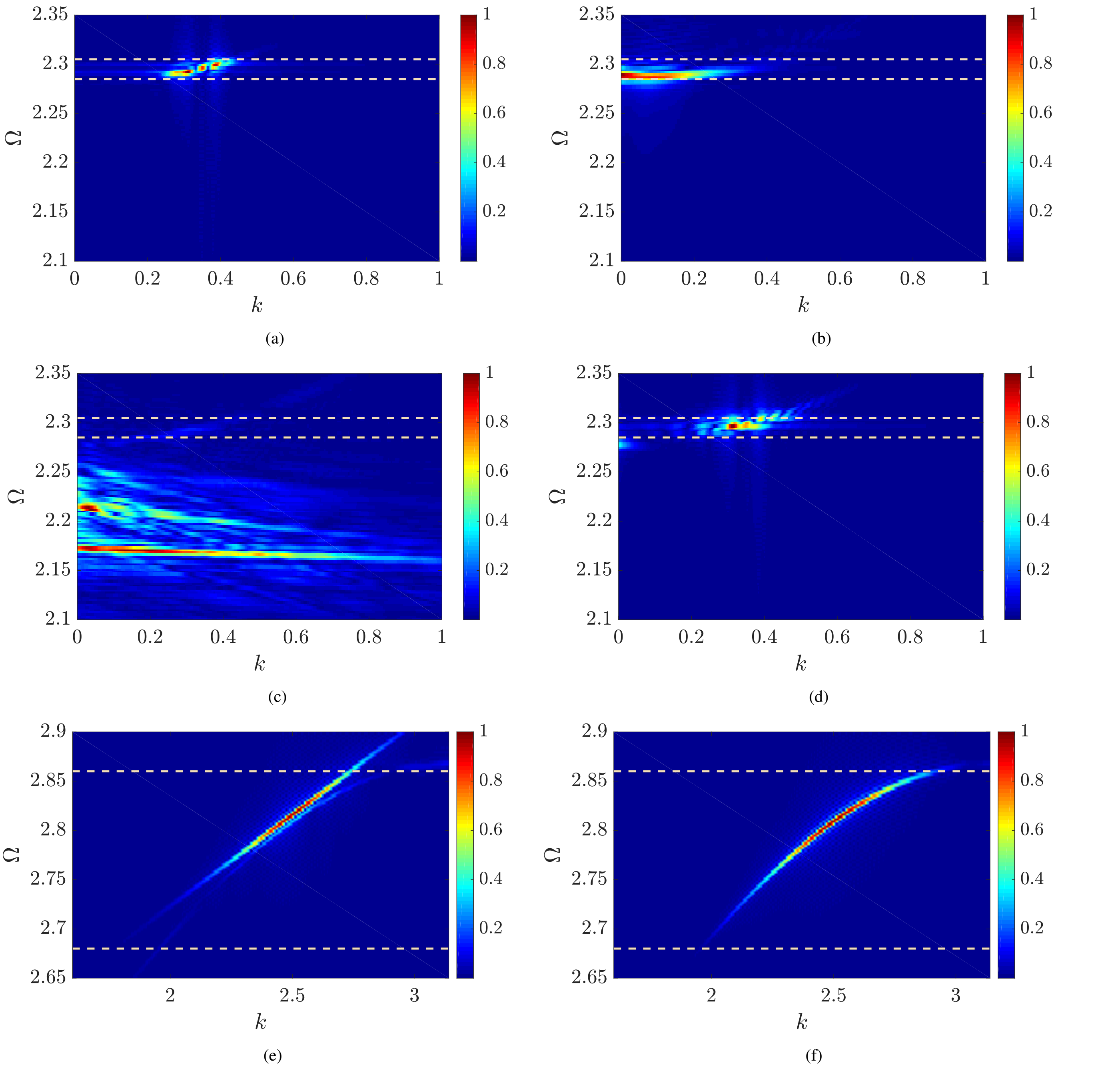

FIGURE 8: 2-D Fourier transform of the response for different types and sources of nonlinearities at frequencies in the upper branch of dispersion curve: (a) Hardening chain nonlinearity $\varepsilon\bar{\Gamma}\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}_2\alpha^2 = 0$, $k = \pi/9$; (b) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = \pi/9$; (c) Softening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = -0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = \pi/9$; (d) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_1\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_2\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = \pi/9$; (e) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_2\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_1\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = 7\pi/9$; (f) Hardening resonator nonlinearity $\varepsilon\bar{\Gamma}_1\alpha^2 = 0.03$, $\varepsilon\bar{\Gamma}_2\alpha^2 = \varepsilon\bar{\Gamma}\alpha^2 = 0$, $k = 7\pi/9$. Dashed lines represents linear frequency bands.

is near the resonator frequency. Hence suggesting that nonlinear resonators, unlike nonlinear chains, affect wave propagation in the long wavelength domain. This observation was consistent with the topological analysis. In the spectro-spatial analysis, we demonstrated that the effect of hardening nonlinearity appears as localizing the wave, whereas, that of softening nonlinearity appears as dispersing the wave. This effect depends on the nonlinear resonator frequency and how close it is to the input wave frequency. Spectrograms and images of 2-D short term Fourier transform also confirmed these observations. They also showed that the nonlinear resonator has output signal stretching over wider range of frequencies and wavenumbers in the long wavelength region. In addition, the nonlinear resonators and nonlinear chains exhibited similar waveform characteristics in short wavelength region when the nonlinear resonator was tuned properly. These observations suggest that such a nonlinear metamaterial, consisting of nonlinear (or linear) chain and multiple linear (or nonlinear) resonators, are suitable for various applications including acoustic diodes and broadband vibration isolation.

ACKNOWLEDGMENT

This work was supported by the start-up grant provided by the Department of Mechanical Engineering at Virginia Tech.

REFERENCES

[1] Hussein, M. I., Leamy, M. J., and Ruzzene, M., 2014. "Dynamics of phononic materials and structures: Historical origins, recent progress, and future outlook". *Applied Mechanics Reviews,* ***66***(4), p. 040802.

[2] Bertoldi, K., Vitelli, V., Christensen, J., and van Hecke, M., 2017. "Flexible mechanical metamaterials". *Nature Reviews Materials,* ***2***(11), p. 17066.

[3] Sigalas, M. M., and Economou, E. N., 1992. "Elastic and acoustic wave band structure". *Journal of Sound Vibration,* ***158***, pp. 377–382.

[4] Sigalas, M., and Economou, E. N., 1993. "Band structure of elastic waves in two dimensional systems". *Solid state communications,* ***86***(3), pp. 141–143.

[5] Kushwaha, M. S., Halevi, P., Dobrzynski, L., and Djafari-Rouhani, B., 1993. "Acoustic band structure of periodic elastic composites". *Physical review letters,* ***71***(13), p. 2022.

[6] Kushwaha, M. S., Halevi, P., Martinez, G., Dobrzynski, L., and Djafari-Rouhani, B., 1994. "Theory of acoustic band structure of periodic elastic composites". *Physical Review B,* ***49***(4), p. 2313.

[7] Vasseur, J., Djafari-Rouhani, B., Dobrzynski, L., Kushwaha, M., and Halevi, P., 1994. "Complete acoustic band gaps in periodic fibre reinforced composite materials: the carbon/epoxy composite and some metallic systems". *Journal of Physics: Condensed Matter,* ***6***(42), p. 8759.

[8] Kushwaha, M. S., 1996. "Classical band structure of periodic elastic composites". *International Journal of Modern Physics B,* ***10***(09), pp. 977–1094.

[9] Liu, Z., Zhang, X., Mao, Y., Zhu, Y., Yang, Z., Chan, C. T., and Sheng, P., 2000. "Locally resonant sonic materials". *science,* ***289***(5485), pp. 1734–1736.

[10] Liu, L., and Hussein, M. I., 2012. "Wave motion in periodic flexural beams and characterization of the transition between bragg scattering and local resonance". *Journal of Applied Mechanics,* ***79***(1), p. 011003.

[11] Huang, G., and Sun, C., 2010. "Band gaps in a multiresonator acoustic metamaterial". *Journal of Vibration and Acoustics,* ***132***(3), p. 031003.

[12] Zhu, R., Liu, X., Hu, G., Sun, C., and Huang, G., 2014. "A chiral elastic metamaterial beam for broadband vibration suppression". *Journal of Sound and Vibration,* ***333***(10), pp. 2759–2773.

[13] Kivshar, Y. S., and Flytzanis, N., 1992. "Gap solitons in diatomic lattices". *Physical Review A,* ***46***(12), p. 7972.

[14] Nadkarni, N., Daraio, C., and Kochmann, D. M., 2014. "Dynamics of periodic mechanical structures containing bistable elastic elements: From elastic to solitary wave propagation". *Physical Review E,* ***90***(2), p. 023204.

[15] Liang, B., Yuan, B., and Cheng, J.-c., 2009. "Acoustic diode: Rectification of acoustic energy flux in one-dimensional systems". *Physical review letters,* ***103***(10), p. 104301.

[16] Manimala, J. M., and Sun, C., 2016. "Numerical investigation of amplitude-dependent dynamic response in acoustic metamaterials with nonlinear oscillators". *The Journal of the Acoustical Society of America,* ***139***(6), pp. 3365–3372.

[17] Nayfeh, A. H., 2011. *Introduction to perturbation techniques*. John Wiley & Sons.

[18] Nayfeh, A. H., and Mook, D. T., 2008. *Nonlinear oscillations*. John Wiley & Sons.

[19] Narisetti, R. K., Leamy, M. J., and Ruzzene, M., 2010. "A perturbation approach for predicting wave propagation in one-dimensional nonlinear periodic structures". *Journal of Vibration and Acoustics,* ***132***(3), p. 031001.

[20] Manktelow, K., Leamy, M. J., and Ruzzene, M., 2011. "Multiple scales analysis of wave–wave interactions in a cubically nonlinear monoatomic chain". *Nonlinear Dynamics,* ***63***(1-2), pp. 193–203.

[21] Liang, B., Guo, X., Tu, J., Zhang, D., and Cheng, J., 2010. "An acoustic rectifier". *Nature materials,* ***9***(12), p. 989.

[22] Li, X.-F., Ni, X., Feng, L., Lu, M.-H., He, C., and Chen, Y.-F., 2011. "Tunable unidirectional sound propagation through a sonic-crystal-based acoustic diode". *Physical review letters,* ***106***(8), p. 084301.

[23] Boechler, N., Theocharis, G., and Daraio, C., 2011.

"Bifurcation-based acoustic switching and rectification". *Nature materials,* ***10***(9), p. 665.
[24] Moore, K. J., Bunyan, J., Tawfick, S., Gendelman, O. V., Li, S., Leamy, M., and Vakakis, A. F., 2018. "Nonreciprocity in the dynamics of coupled oscillators with nonlinearity, asymmetry, and scale hierarchy". *Physical Review E,* ***97***(1), p. 012219.
[25] Ma, C., Parker, R. G., and Yellen, B. B., 2013. "Optimization of an acoustic rectifier for uni-directional wave propagation in periodic mass–spring lattices". *Journal of Sound and Vibration,* ***332***(20), pp. 4876–4894.
[26] Ganesh, R., and Gonella, S., 2013. "Spectro-spatial wave features as detectors and classifiers of nonlinearity in periodic chains". *Wave Motion,* ***50***(4), pp. 821–835.
[27] Zhou, W., Li, X., Wang, Y., Chen, W., and Huang, G., 2018. "Spectro-spatial analysis of wave packet propagation in nonlinear acoustic metamaterials". *Journal of Sound and Vibration,* ***413***, pp. 250–269.
[28] Bukhari, M. A., and Barry, O. R., 2019. "Nonlinear metamaterials with multiple local mechanical resonators: Analytical and numerical analyses". In NODYCON 2019 The First International Nonlinear Dynamics Conference, (Submitted).
[29] Manimala, J. M., 2014. "Dynamic behavior of acoustic metamaterials and metaconfigured structures with local oscillators". PhD thesis, Purdue University.